

2x2 Matrix: Calculation of the n-th Power

Abstract

This work explores a robust method for calculating the n-th power of a 2x2 matrix A^n without using diagonalization techniques. This approach significantly simplifies complex matrix operations. Firstly, we give here, a single formula for the n th power for any type of 2x2 matrix, that is to say whether it is diagonalizable or not, whether it has two distinct eigenvalues or not and whether these eigenvalues are real or not. We also add a second method.

Keywords: Eigenvalue, The nth power of matrices, The second-order linear recurrent sequence, The characteristic polynomial

2010 Mathematics Subject Classification: 15A15; 15A16; 15A18

1 Introduction

In 1992 *Kenneth S. Williams* (see (?)) gives us a general formula working with the Matrices $X = \frac{A-\beta I}{\alpha-\beta}$, $Y = \frac{A-\alpha I}{\beta-\alpha}$ et $Z = A - \alpha I$, which is

$$A^n = \begin{cases} \alpha^n \left(\frac{A-\beta I}{\alpha-\beta} \right) + \beta^n \left(\frac{A-\alpha I}{\beta-\alpha} \right) & \text{if } \alpha \neq \beta \\ \alpha^{n-1} (nA - (n-1)\alpha I) & \text{if } \alpha = \beta \end{cases}$$

In the article (?), they study and determine a general formula for the nth power of 2x2 matrices using the eigenvalues of A, and give some Excel programming examples.

Here we give, and this is new, a single formula for the nth power for any type of 2x2 matrix, that is to say whether it is diagonalizable or not, whether it has two distinct eigenvalues or not and whether these eigenvalues are real or not. We also add a second method.

2 The power of any (diagonalizable or not) matrix 2x2

Theorem 2.1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a 2×2 matrix, so $\forall n \geq 2$ ($n \in \mathbb{N}$)

$$A^n = \begin{pmatrix} a \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k} & b \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} \\ c \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} & \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} - a \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} \end{pmatrix}$$

with α_1 and α_2 solutions of $|A - \alpha I| = 0$.

Proof. We will demonstrate our formula using recurrence.

(i) For $n = 2$, we will show that we actually have

$$A^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}$$

$$A^2 = \begin{pmatrix} a \sum_{k=0}^1 \alpha_1^k \alpha_2^{1-k} - \alpha_1 \alpha_2 & b \sum_{k=0}^1 \alpha_1^k \alpha_2^{1-k} \\ c \sum_{k=0}^1 \alpha_1^k \alpha_2^{1-k} & \sum_{k=0}^2 \alpha_1^k \alpha_2^{2-k} - a \sum_{k=0}^1 \alpha_1^k \alpha_2^{1-k} \end{pmatrix}$$

we must therefore show the following 4 equalities

$$\begin{cases} a^2 + bc = a \sum_{k=0}^1 \alpha_1^k \alpha_2^{1-k} - \alpha_1 \alpha_2 \\ d^2 + bc = \sum_{k=0}^2 \alpha_1^k \alpha_2^{2-k} - a \sum_{k=0}^1 \alpha_1^k \alpha_2^{1-k} \\ b(a+d) = b \sum_{k=0}^1 \alpha_1^k \alpha_2^{1-k} \\ c(a+d) = c \sum_{k=0}^1 \alpha_1^k \alpha_2^{1-k} \end{cases}$$

what does it mean

$$\begin{cases} a^2 + bc = a(\alpha_2 + \alpha_1) - \alpha_1 \alpha_2 \\ d^2 + bc = \alpha_2^2 + \alpha_1 \alpha_2 + \alpha_1^2 - a(\alpha_2 + \alpha_1) \\ b(a+d) = b(\alpha_2 + \alpha_1) \\ c(a+d) = c(\alpha_2 + \alpha_1) \end{cases}$$

α_1 and α_2 solutions of $|A - \alpha I| = 0$, so

$$\begin{cases} \alpha_1 + \alpha_2 = \text{Tr}(A) = a + d \\ \alpha_1 \alpha_2 = |A| = ad - bc \end{cases}$$

So

$$a(\alpha_2 + \alpha_1) - \alpha_1 \alpha_2 = a(a+d) - (ad - bc) = a^2 + ad - ad + bc = a^2 + bc$$

$$\alpha_2^2 + \alpha_1 \alpha_2 + \alpha_1^2 - a(\alpha_2 + \alpha_1) = (\alpha_1 + \alpha_2)^2 - \alpha_1 \alpha_2 - a(\alpha_2 + \alpha_1) \implies$$

$$\alpha_2^2 + \alpha_1 \alpha_2 + \alpha_1^2 - a(\alpha_2 + \alpha_1) = (a+d)^2 - (ad - bc) - a(a+d) = d^2 + bc$$

$$b(\alpha_2 + \alpha_1) = b(a+d)$$

$$c(\alpha_2 + \alpha_1) = c(a+d)$$

hence the result is true for $n = 2$.

(ii) Let's suppose

$$A^n = \begin{pmatrix} a \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k} & b \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} \\ c \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} & \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} - a \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} \end{pmatrix}$$

(iii) Let us show that

$$A^{n+1} = A^n . A$$

$$A^{n+1} = \begin{pmatrix} a \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} - \sum_{k=0}^{n-1} \alpha_1^{k+1} \alpha_2^{n-k} & b \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} \\ c \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} & \sum_{k=0}^{n+1} \alpha_1^k \alpha_2^{n+1-k} - a \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} \end{pmatrix}$$

We have $A^{n+1} = A^n \cdot A =$

$$= \begin{pmatrix} a \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k} & b \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} \\ c \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} & \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} - a \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} \end{pmatrix} A$$

$$A^{n+1} = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

$$x = a^2 \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - a \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k} + cb \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k}$$

$$y = ba \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - b \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k} + db \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k}$$

$$z = ac \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} + c \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} - ca \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k}$$

$$t = bc \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} + d \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} - da \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k}$$

$$\Rightarrow \begin{cases} x = (a^2 + cb) \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - a \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k} \\ y = b \left[(a + d) \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k} \right] \\ z = c \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} \\ t = (bc - da) \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} + d \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} \end{cases}$$

\Rightarrow

$$x = [a(\alpha_1 + \alpha_2) - \alpha_1 \alpha_2] \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - a \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k}$$

$$y = b \left[(\alpha_1 + \alpha_2) \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k} \right]$$

$$z = c \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k}$$

$$t = -\alpha_1 \alpha_2 \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} + d \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k}$$

because $\alpha_1 + \alpha_2 = a + d$, $\alpha_1 \alpha_2 = ad - bc$, $a(\alpha_1 + \alpha_2) - \alpha_1 \alpha_2 = a^2 + cb$

$$A^{n+1} = \begin{pmatrix} a \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} - \alpha_1 \alpha_2 \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} & b \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} \\ c \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} & \sum_{k=0}^{n+1} \alpha_1^k \alpha_2^{n+1-k} - a \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} \end{pmatrix}$$

because

$$x = [a(\alpha_1 + \alpha_2) - \alpha_1 \alpha_2] \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - a \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k}$$

$$= a(\alpha_1 + \alpha_2) \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - \alpha_1 \alpha_2 \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - a \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k}$$

$$= a\alpha_1 \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} + a\alpha_2 \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - \alpha_1 \alpha_2 \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - a \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k}$$

$$= a\alpha_1^n + a \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-k} - \alpha_1 \alpha_2 \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k}$$

$$x = a \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} - \alpha_1 \alpha_2 \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} ;$$

$$\begin{aligned}
 y &= b \left[(\alpha_1 + \alpha_2) \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k} \right] \\
 y &= b \left[\sum_{k=0}^{n-1} \alpha_1^{k+1} \alpha_2^{n-1-k} + \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-k} - \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k} \right] \\
 y &= b \left[\alpha_1^n + \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-k} \right] = b \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} \\
 \text{and} \\
 t &= -\alpha_1 \alpha_2 \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} + d \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} \\
 t &= -\sum_{k=0}^{n-1} \alpha_1^{k+1} \alpha_2^{n-k} + (\alpha_1 + \alpha_2 - a) \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} \text{ (as } \alpha_1 + \alpha_2 = a + d) \\
 t &= -\sum_{k=0}^{n-1} \alpha_1^{k+1} \alpha_2^{n-k} - a \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} + \sum_{k=0}^n \alpha_1^{k+1} \alpha_2^{n-k} + \sum_{k=0}^n \alpha_1^k \alpha_2^{n+1-k} \\
 t &= -a \sum_{k=0}^n \alpha_1^k \alpha_2^{n-k} + \sum_{k=0}^{n+1} \alpha_1^k \alpha_2^{n+1-k}
 \end{aligned}$$

hence the result. □

Remark 2.1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a 2×2 matrix and α_1 and α_2 solutions of $|A - \alpha I| = 0$.

1) If $\alpha_1 = \alpha_2 = \alpha$, so

$$A^n = \left(\frac{a+d}{2} \right)^{n-1} \begin{pmatrix} \frac{(n+1)a - (n-1)d}{2} & nb \\ nc & \frac{(n+1)d - (n-1)a}{2} \end{pmatrix}, \forall n \geq 2 \ (n \in \mathbb{N})$$

with $\alpha = \frac{a+d}{2}$ and $(a+d)^2 = 4(ad-bc)$ ($(a-d)^2 = -4bc$)

And

$$A^n = n\alpha^{n-1}A - (n-1)\alpha^n I$$

2) If $\alpha_1 \neq \alpha_2$ so $\forall n \geq 2 \ (n \in \mathbb{N})$

$$\begin{aligned}
 A^n &= \frac{1}{\alpha_1 - \alpha_2} \begin{pmatrix} (a - \alpha_2)\alpha_1^n - (a - \alpha_1)\alpha_2^n & b(\alpha_1^n - \alpha_2^n) \\ c(\alpha_1^n - \alpha_2^n) & (a - \alpha_2)\alpha_2^n - (a - \alpha_1)\alpha_1^n \end{pmatrix} \\
 A^n - \alpha_2^n I &= \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} (A - \alpha_2 I) \text{ and } A^n - \alpha_1^n I = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} (A - \alpha_1 I) \\
 A^n &= \left(\frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} \right) A - \alpha_1 \alpha_2 \left(\frac{\alpha_1^{n-1} - \alpha_2^{n-1}}{\alpha_1 - \alpha_2} \right) I
 \end{aligned}$$

Proof. 1) $A^n = \alpha^{n-1} \begin{pmatrix} na - (n-1)\alpha & nb \\ nc & (n+1)\alpha - na \end{pmatrix}$

And

$$\begin{aligned}
 A^n &= \alpha^{n-1} \begin{pmatrix} na - (n-1)\alpha & nb \\ nc & nd + (n+1)\alpha - n(a+d) \end{pmatrix} \\
 A^n &= \alpha^{n-1} \begin{pmatrix} na - (n-1)\alpha & nb \\ nc & nd + (n+1)\alpha - 2n\alpha \end{pmatrix} \\
 A^n &= \alpha^{n-1} \begin{pmatrix} na - (n-1)\alpha & nb \\ nc & nd + (1-n)\alpha \end{pmatrix} \\
 A^n &= \alpha^{n-1} \begin{pmatrix} na & nb \\ nc & nd \end{pmatrix} + \alpha^{n-1} \begin{pmatrix} -(n-1)\alpha & 0 \\ 0 & -(n-1)\alpha \end{pmatrix}
 \end{aligned}$$

$$A^n = n\alpha^{n-1}A - (n-1)\alpha^n I$$

2)

$$A^n = \frac{1}{\alpha_1 - \alpha_2} \begin{pmatrix} a(\alpha_1^n - \alpha_2^n) - \alpha_1\alpha_2(\alpha_1^{n-1} - \alpha_2^{n-1}) & b(\alpha_1^n - \alpha_2^n) \\ c(\alpha_1^n - \alpha_2^n) & (\alpha_1^{n+1} - \alpha_2^{n+1}) - a(\alpha_1^n - \alpha_2^n) \end{pmatrix}$$

□

2.1 Second method:

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$ a 2X2 matrix, so $|A - xI| = x^2 - \text{Tr}(A)x + |A|$

with $\text{Tr}(A) = a + d \in \mathbb{R}$ and $|A| = ad - bc \in \mathbb{R}$

$$\implies A^2 - \text{Tr}(A)A + |A|I = O$$

$$\implies A^2 = \text{Tr}(A)A - |A|I$$

$$\forall A \in M_2(\mathbb{R}), \exists (r, s) \in \mathbb{R}^2 / A^2 = rA + sI$$

\implies

$$A^n = r_n A + s_n I, \forall n \in \mathbb{N}$$

so $A^{n+1} = r_{n+1}A + s_{n+1}I = r_n A^2 + s_n A$

$$\implies r_{n+1}A + s_{n+1}I = r_n(rA + sI) + s_n A = (rr_n + s_n)A + sr_n I$$

\implies

$$\begin{cases} r_{n+1} = rr_n + s_n \\ s_{n+1} = sr_n \end{cases}$$

\implies

$$\begin{pmatrix} r_{n+1} \\ s_{n+1} \end{pmatrix} = \begin{pmatrix} r & 1 \\ s & 0 \end{pmatrix} \begin{pmatrix} r_n \\ s_n \end{pmatrix}$$

\implies

$$\begin{cases} r_{n+1} = rr_n + sr_{n-1} \\ s_{n+1} = sr_n \end{cases}$$

with $r_1 = 1, r_2 = r, s_1 = 0$ and $s_2 = s$.

Remark 2.2.

$$(r_n, s_n) \in \mathbb{R}^2, \forall n \in \mathbb{N}^*.$$

Proof. By recurrence:

We have $(r_2, s_2) = (r, s) \in \mathbb{R}^2$

$$\text{and } \begin{pmatrix} r_{n+1} \\ s_{n+1} \end{pmatrix} = \begin{pmatrix} r & 1 \\ s & 0 \end{pmatrix} \begin{pmatrix} r_n \\ s_n \end{pmatrix}.$$

□

First we will solve the second-order linear recurrent sequence with constant coefficients: $r_{n+1} = rr_n + sr_{n-1}$

the characteristic equation is

$$x^2 = rx + s$$

$$P(x) = x^2 - rx - s = 0$$

which (P) is the same as the characteristic polynomial of A

$$x = \frac{r \pm (r^2 + 4s)^{\frac{1}{2}}}{2} = \begin{cases} \alpha_1 \\ \alpha_2 \end{cases}$$

$$(\alpha_1, \alpha_2) \in \mathbb{C}^2$$

$$\implies \begin{cases} r_n = c_1 \alpha_1^n + c_2 \alpha_2^n & \text{if } \alpha_1 \neq \alpha_2 \\ \text{and} \\ r_n = (c_1 n + c_2) \alpha^n & \text{if } \alpha_1 = \alpha_2 = \alpha \end{cases}$$

2.1.1 If $\alpha_1 \neq \alpha_2$:

We have

$$r_1 = c_1 \alpha_1 + c_2 \alpha_2 = 1, r_2 = c_1 \alpha_1^2 + c_2 \alpha_2^2 = r$$

so,

$$\begin{cases} c_1 = \frac{r - \alpha_2}{\alpha_1(\alpha_1 - \alpha_2)} \\ c_2 = \frac{r - \alpha_1}{\alpha_2(\alpha_2 - \alpha_1)} \end{cases} \quad \text{if } \alpha_1 \neq \alpha_2$$

$$r_n = \frac{1}{\alpha_1 - \alpha_2} [(r - \alpha_2) \alpha_1^{n-1} - (r - \alpha_1) \alpha_2^{n-1}] \quad \text{if } \alpha_1 \neq \alpha_2$$

$$r_n = \frac{1}{\alpha_1 - \alpha_2} [r(\alpha_1^{n-1} - \alpha_2^{n-1}) - \alpha_1 \alpha_2 (\alpha_1^{n-2} - \alpha_2^{n-2})]$$

and

$$\begin{aligned} s_n &= sr_{n-1} = \frac{s}{\alpha_1 - \alpha_2} [(r - \alpha_2) \alpha_1^{n-2} - (r - \alpha_1) \alpha_2^{n-2}] \\ &= \frac{s}{\alpha_1 - \alpha_2} [r(\alpha_1^{n-2} - \alpha_2^{n-2}) - \alpha_1 \alpha_2 (\alpha_1^{n-3} - \alpha_2^{n-3})] \end{aligned}$$

So

$$(\alpha_1 - \alpha_2) A^n = [(r - \alpha_2) \alpha_1^{n-1} - (r - \alpha_1) \alpha_2^{n-1}] A + s [(r - \alpha_2) \alpha_1^{n-2} - (r - \alpha_1) \alpha_2^{n-2}] I$$

Since α_1 and α_2 are solutions of the equation $x^2 = rx + s$

$$\begin{aligned} \implies \alpha_1^2 &= r\alpha_1 + s, \alpha_2^2 = r\alpha_2 + s \text{ and } s = -\alpha_1 \alpha_2 \\ \implies \alpha_1^n &= r\alpha_1^{n-1} + s\alpha_1^{n-2}, \alpha_2^n = r\alpha_2^{n-1} + s\alpha_2^{n-2} \\ \implies \alpha_1^n - \alpha_2^n &= r(\alpha_1^{n-1} - \alpha_2^{n-1}) - \alpha_1 \alpha_2 (\alpha_1^{n-2} - \alpha_2^{n-2}) \\ \implies \alpha_1^n - \alpha_2^n &= (r - \alpha_2) \alpha_1^{n-1} - (r - \alpha_1) \alpha_2^{n-1} \\ \text{and } \alpha_1^{n-1} - \alpha_2^{n-1} &= (r - \alpha_2) \alpha_1^{n-2} - (r - \alpha_1) \alpha_2^{n-2} \\ \implies \end{aligned}$$

$$\begin{aligned} A^n &= \left(\frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} \right) A - \alpha_1 \alpha_2 \left(\frac{\alpha_1^{n-1} - \alpha_2^{n-1}}{\alpha_1 - \alpha_2} \right) I \\ A^n &= \left(\frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} \right) A + s \left(\frac{\alpha_1^{n-1} - \alpha_2^{n-1}}{\alpha_1 - \alpha_2} \right) I \end{aligned}$$

2.1.2 If $\alpha_1 = \alpha_2 = \alpha \neq 0$:

so $\alpha_1 = \alpha_2 = \alpha = \frac{r}{2}$ (since $\Delta = 0$)

$$r_n = (c_1 n + c_2) \alpha^n$$

$$r_1 = (c_1 + c_2) \alpha = 1, r_2 = (2c_1 + c_2) \alpha^2 = r$$

\implies

$$\begin{cases} c_1 = \frac{r - \alpha}{2\alpha^2} \\ c_2 = \frac{2\alpha - r}{\alpha^2} \end{cases} \quad \text{if } \alpha_1 = \alpha_2$$

\implies

$$r_n = [(r - \alpha)n + r - 2\alpha] \alpha^{n-2}$$

And $s_{n+1} = sr_n \implies s_n = s [(r - \alpha)(n - 1) + r - 2\alpha] \alpha^{n-3}$
 $\implies s_n = s [(r - \alpha)n - \alpha] \alpha^{n-3}$
 $\implies A^n = [(r - \alpha)n + r - 2\alpha] \alpha^{n-2} A + s [(r - \alpha)n - \alpha] \alpha^{n-3} I$
 $\alpha = \frac{r}{2}$ and $\implies \alpha^2 = -s$

$$A^n = n \left(\frac{r}{2}\right)^{n-1} A + s(n-1) \left(\frac{r}{2}\right)^{n-2} I$$

$$A^n = n\alpha^{n-1} A - (n-1)\alpha^n I$$

If $\alpha_1 = \alpha_2 = 0$
 so $A^2 = O$ and $A^n = O, \forall n \geq 2$.

Remark 2.3. 1) If $\alpha_1 = \alpha e^{i\theta} \in \mathbb{C} \setminus \mathbb{R}$, so $\alpha_2 = \bar{\alpha}_1 = \alpha e^{-i\theta} \in \mathbb{C} \setminus \mathbb{R}$ and

$$A^n = \frac{\alpha^{n-2}}{\sin(\theta)} \{ \alpha \sin(n\theta) A + s \sin[(n-1)\theta] I \}$$

2) We can do the same if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C})$.

Conclusion 2.2. *We conclude that the final formulas are the same as those given in ? and ? if we distinguish the cases according to whether the eigenvalues are equal or not, except that in our article we were able to combine all the cases in a single formula, but for the second method it was done only in the case where the matrix has real coefficients, but this still gives us the credibility that our formulas and calculations are indeed true.*

References

Williams, K. S. (1992). The n th power of a 2×2 matrix (in Notes), Math. Magazine Vol.65, No.5, p. 336.

Jung Hyeon Ryu and Won Kyu Kim (2015). A formula on power n of 2×2 Matrices. Far East Journal of Mathematical Sciences (FJMS). http://dx.doi.org/10.17654/FJMSFeb2015_365_377. Volume 96, Issue 3, Pages 365 - 377.