Medical SuperHyperStructure and Healthcare SuperHyperStructure

Abstract

We develop a rigorous, set–theoretic framework for modeling clinical and health-system decision making using *Hyperstructures* and their iterated counterparts, *Superhyperstructures*. On the clinical side, we formalize a *Medical HyperStructure* on a space of patient states equipped with (i) a valuation into a partially ordered commutative monoid V capturing risk, cost, workload, and related quantities; (ii) policy/feasibility correspondences that encode guidelines and contraindications; (iii) observation maps for measurable summaries; and (iv) convex-cone bounds on admissible changes in valuations. Lifting these ingredients to iterated powersets yields *Medical SuperHyperStructures of order* (m, n), which capture cohorts, multi-stage care pathways, and protocol families via set-valued outputs and lifted observations.

On the operational side, we analogously define *Healthcare Hyper/SuperHyperStructures* on system states (queues, capacities, rosters, inventories) with KPI vectors and policy constraints. The framework unifies many rule-based and protocol-driven processes as special cases—e.g., $(m, n) = (0, 1)$ recovers classical hyperoperations—while supporting uncertainty, aggregation across levels, and resource/safety budgets through convex-cone admissibility. The paper is purely theoretical (no data or simulation). Worked examples—empiric antibiotics, insulin titration, neuroimaging triage, ED admission policy, OR day scheduling, inventory replenishment, and outpatient overbooking—illustrate expressiveness and suggest directions for quantitative validation in future studies.

Keywords: HyperStructure, SuperHyperStructure, Medical SuperHyperStructure, Healthcare SuperHyperStructure

1 Preliminaries

This section gathers the core notions and notation used throughout the paper. Unless explicitly stated otherwise, all underlying sets are *finite*. By convention, we also regard the empty set $\emptyset$ as an element of every set.

1.1 Hyperstructure and Superhyperstructure

A *Hyperstructure* arises from the powerset construction and provides a general framework for modeling relations among elements of a set [1–4]. Thanks to its flexibility, the hyperstructure paradigm has been applied widely, including in mathematics and chemistry [5–7]. Extending this idea, a *Superhyperstructure* employs the n-th iterated powerset to encode multi-layered hierarchical relationships, enabling deeper abstraction and greater structural complexity [8, 9]. Because of its broad applicability, the superhyperstructure viewpoint has likewise been studied in mathematics, chemistry, and related areas [10, 11]. For reference, we record below the basic set-theoretic constructions on which these frameworks rest.

Definition 1.1 (Base Set)**.** A *base set* S is the underlying collection of elements from which more elaborate constructions—such as powersets and hyperstructures—are formed. Formally,

$$S \;=\; \{\, x \mid x \text{ belongs to a specified domain} \,\}.$$

All elements of constructions like $\mathcal{P}(S)$ and the iterated powersets $\mathcal{P}_n(S)$ ultimately originate from the elements of S.

Definition 1.2 (Powerset)**.** [12] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the family of all subsets of S, including both $\emptyset$ and S itself:

$$\mathcal{P}(S) \;=\; \{\, A \mid A \subseteq S \,\}.$$

Definition 1.3 (n-th Powerset)**.** (cf. [9, 13]) For a set H, the n-th powerset $\mathcal{P}_n(H)$ is defined recursively by

$$\mathcal{P}_1(H) = \mathcal{P}(H), \qquad \mathcal{P}_{n+1}(H) = \mathcal{P}\big(\mathcal{P}_n(H)\big) \quad (n \geq 1).$$

The *nonempty* n-th powerset, denoted $\mathcal{P}^*_n(H)$, is obtained analogously by removing the empty set at each stage:

$$\mathcal{P}^*_1(H) = \mathcal{P}^*(H), \qquad \mathcal{P}^*_{n+1}(H) = \mathcal{P}^*\big(\mathcal{P}^*_n(H)\big),$$

where $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$.

Example 1.4 (3rd Powerset: Weekly meal plans from dish choices)**.** Let the base set of dishes be

$$H = \{\text{Salad}, \ \text{Soup}, \ \text{Pasta}, \ \text{Curry}\}.$$

Level 1 ($\mathcal{P}(H)$): Menus. Typical menus (subsets of dishes) are

$$M_1 = \{\text{Salad}, \text{Pasta}\}, \quad M_2 = \{\text{Soup}, \text{Curry}\}, \quad M_3 = \{\text{Salad}, \text{Soup}, \text{Curry}\} \;\in\; \mathcal{P}(H).$$

Level 2 ($\mathcal{P}^2(H) = \mathcal{P}(\mathcal{P}(H))$): Day plan options. A *day plan* is a set of admissible menus, e.g.

$$D_{\text{Mon}} = \{M_1, M_2\}, \qquad D_{\text{Tue}} = \{M_2, M_3\} \;\in\; \mathcal{P}^2(H).$$

Level 3 ($\mathcal{P}^3(H) = \mathcal{P}(\mathcal{P}^2(H))$): Weekly plan families. A *weekly plan family* collects possible day plans, e.g.

$$W_\alpha = \{D_{\text{Mon}}, D_{\text{Tue}}\}, \qquad W_\beta = \{D_{\text{Tue}}\} \in \mathcal{P}^3(H).$$

Thus elements of $\mathcal{P}^3(H)$ (like W_α, W_β) are sets of day-plan *sets* of menus, providing a three-tier representation:

dishes $\rightarrow$ menus $\rightarrow$ day plans $\rightarrow$ weekly plan families.

If one wishes to forbid empties at each tier, replace $\mathcal{P}$ by $\mathcal{P}^*(\cdot) = \mathcal{P}(\cdot) \setminus \{\emptyset\}$ throughout to obtain $\mathcal{P}^{*3}(H)$.

Example 1.5 (Reaction-screening design as tiers of sets ($n = 1, 2, 3$))**.** Let the base set of *bench-top items*

$$H = \{\text{AcOH}, \ \text{EtOH}, \ \text{H}_2\text{SO}_4, \ \text{sieve}\}$$

collect acetic acid, ethanol, sulfuric acid catalyst, and molecular sieve (drying agent). Level 1 ($\mathcal{P}(H)$) consists of candidate *mixtures*. For instance,

$$M_1 = \{\text{AcOH}, \text{EtOH}, \text{H}_2\text{SO}_4\}, \qquad M_2 = \{\text{AcOH}, \text{EtOH}, \text{H}_2\text{SO}_4, \text{sieve}\} \in \mathcal{P}(H).$$

Level 2 ($\mathcal{P}^2(H) = \mathcal{P}(\mathcal{P}(H))$) collects *parallel batches/plates* as sets of mixtures:

$$D = \{M_1, M_2\} \in \mathcal{P}^2(H).$$

Level 3 ($\mathcal{P}^3(H)$) groups such plates into *campaigns*:

$$W = \{D, \ \{M_1\}\} \in \mathcal{P}^3(H).$$

Cardinalities (including empties) illustrate combinatorial growth: $|H| = 4 \Rightarrow |\mathcal{P}(H)| = 2^4 = 16$, hence $|\mathcal{P}^2(H)| = 2^{16} = 65{,}536$, and $|\mathcal{P}^3(H)| = 2^{65{,}536}$. If empties are excluded at each tier, replace $\mathcal{P}$ by $\mathcal{P}^*$, so $|\mathcal{P}_1^*(H)| = 2^4 - 1 = 15$, and generally $\mathcal{P}_{n+1}^*(H) = \mathcal{P}^*(\mathcal{P}_n^*(H))$.

Example 1.6 (Mechanistic portfolios for acid-catalyzed dehydration ($n = 1, 2$))**.** Let the base set of *elementary steps*

$$H = \{\text{protonation}, \ \text{C–O cleavage}, \ \text{hydride shift}, \ \text{deprotonation}\}.$$

Level 1: a *mechanism* is a subset of steps, e.g.

$$\begin{aligned} M_{\text{EtOH}} &= \{\text{protonation}, \ \text{C–O cleavage}, \ \text{deprotonation}\}, \\ M_{\text{alt}} &= \{\text{protonation}, \ \text{hydride shift}, \ \text{deprotonation}\} \quad \in \mathcal{P}(H). \end{aligned}$$

Level 2: a *solvent/catalyst portfolio* is a set of mechanisms, $\mathcal{S} = \{M_{\text{EtOH}}, M_{\text{alt}}\} \in \mathcal{P}^2(H)$. This organizes step-level objects (level 1) into families chosen per medium (level 2), and can be extended to level 3 to bundle portfolios across temperature grids or promoter loadings.

To establish a comprehensive framework for understanding Hyperstructures and Superhyperstructures, we present the following formal definitions and foundational concepts.

Definition 1.7 (Classical Structure)**.** (cf. [9,13,14]) A *Classical Structure* is a mathematical framework defined on a non-empty set H, characterized by one or more *Classical Operations* that adhere to specific *Classical Axioms*. Formally:

A *Classical Operation* is a function of the form:

$$\#_0 : H^m \to H,$$

where $m \geq 1$ denotes a positive integer, and H^m represents the m-fold Cartesian product of H. Examples include algebraic operations such as addition and multiplication in structures like groups, rings, and fields.

Definition 1.8 (Hyperoperation)**.** (cf. [8, 15, 16]) A *hyperoperation* is a generalization of a binary operation in which the result of combining two inputs is a *set* (not necessarily a singleton). Formally, for a set S, a hyperoperation $\circ$ is a map

$$\circ : S \times S \longrightarrow \mathcal{P}(S).$$

Definition 1.9 (Hyperstructure)**.** (cf. [9,13,17,18]) A *Hyperstructure* extends the concept of a Classical Structure by operating on the powerset of a base set. It is formally defined as:

$$\mathcal{H} = (\mathcal{P}(S), \circ),$$

where S is the base set, $\mathcal{P}(S)$ denotes its powerset, and $\circ$ is an operation defined for subsets within $\mathcal{P}(S)$.

Example 1.10 (Everyday HyperStructure: Air–travel hub suggestion)**.** Let S be a finite set of airports

$$S = \{\mathrm{HND}, \mathrm{ICN}, \mathrm{SIN}, \mathrm{DOH}, \mathrm{LAX}, \mathrm{JFK}\}.$$

Assume the directed flight network (toy data)

$$\begin{aligned} &\mathrm{HND} \to \{\mathrm{ICN}, \mathrm{SIN}, \mathrm{DOH}\}, \quad \mathrm{ICN} \to \{\mathrm{LAX}, \mathrm{JFK}, \mathrm{HND}\}, \quad \mathrm{SIN} \to \{\mathrm{DOH}, \mathrm{LAX}\}, \\ &\mathrm{DOH} \to \{\mathrm{LAX}, \mathrm{JFK}, \mathrm{HND}\}, \quad \mathrm{LAX} \to \{\mathrm{ICN}, \mathrm{DOH}, \mathrm{HND}\}, \quad \mathrm{JFK} \to \{\mathrm{HND}, \mathrm{DOH}\}. \end{aligned}$$

Define a hyperoperation $\circ : S \times S \to \mathcal{P}(S)$ by

$$a \circ b \;:=\; \{\, h \in S \mid a \to h \text{ and } h \to b \text{ are direct flights} \,\},$$

i.e. $a \circ b$ is the set of feasible *hubs* for a two–hop trip $a \to h \to b$. Then $\mathcal{H} = (\mathcal{P}(S), \circ)$ is a hyperstructure. Concrete evaluations:

$$\mathrm{HND} \circ \mathrm{LAX} = \{\mathrm{ICN}, \mathrm{SIN}, \mathrm{DOH}\}, \qquad \mathrm{HND} \circ \mathrm{JFK} = \{\mathrm{ICN}, \mathrm{DOH}\}.$$

Definition 1.11 (SuperHyperOperation)**.** [9] Let H be a nonempty set. Define recursively, for each integer $k \geq 0$,

$$\mathcal{P}^0(H) = H, \qquad \mathcal{P}^{k+1}(H) = \mathcal{P}\big(\mathcal{P}^k(H)\big).$$

Fix $m, n \geq 0$. An (m,n)-*SuperHyperOperation* is a map

$$\odot^{(m,n)} : \big(\mathcal{P}^m(H)\big)^s \longrightarrow \mathcal{P}^n(H)$$

for some input arity $s \in \mathbb{Z}_{>0}$. When the codomain is allowed to contain $\emptyset$, we speak of the *Neutrosophic* variant; otherwise it is the classical variant.

Definition 1.12 (n-Superhyperstructure)**.** (cf. [9, 13]) An n-*Superhyperstructure* generalizes the Hyperstructure by employing the n-th powerset of a base set. Formally, it is defined as:

$$\mathcal{SH}_n = (\mathcal{P}_n(S), \circ),$$

where S is the base set, $\mathcal{P}_n(S)$ represents the n-th powerset of S, and $\circ$ is an operation acting on elements of $\mathcal{P}_n(S)$.

Example 1.13 (Everyday n-Superhyperstructure ($n = 2$): Round–trip hub families)**.** Let S be the airport set above. Elements of $\mathcal{P}^2(S)$ are *families of hub-sets*. For $A, B \in \mathcal{P}^2(S)$, define

$$A \star^{[2]} B \;:=\; \big\{\, \{h_{\mathrm{out}}, h_{\mathrm{ret}}\} \subseteq S \;\big|\; \exists\, H_{\mathrm{out}} \in A,\ \exists\, H_{\mathrm{ret}} \in B :\ h_{\mathrm{out}} \in H_{\mathrm{out}},\ h_{\mathrm{ret}} \in H_{\mathrm{ret}} \,\big\}.$$

Intuitively, A lists admissible *outbound* hubs and B admissible *return* hubs; the output is the family of round–trip hub pairs. Take, for the toy network,

$$A = \big\{\, \{\mathrm{ICN}, \mathrm{SIN}, \mathrm{DOH}\} \,\big\} \quad (\text{outbound HND} \to \text{LAX}), \qquad B = \big\{\, \{\mathrm{ICN}, \mathrm{DOH}\} \,\big\} \quad (\text{return LAX} \to \text{HND}),$$

so

$$A \star^{[2]} B = \big\{\{\mathrm{ICN}, \mathrm{ICN}\},\ \{\mathrm{ICN}, \mathrm{DOH}\},\ \{\mathrm{SIN}, \mathrm{ICN}\},\ \{\mathrm{SIN}, \mathrm{DOH}\},\ \{\mathrm{DOH}, \mathrm{ICN}\},\ \{\mathrm{DOH}, \mathrm{DOH}\}\big\} \;\in\; \mathcal{P}^2(S).$$

Hence $\mathcal{SH}_2 = (\mathcal{P}^2(S), \star^{[2]})$ is an n-superhyperstructure capturing families of round–trip hub choices.

Definition 1.14 (SuperHyperStructure of order (m,n))**.** (cf. [9, 11, 19]) Let S be a nonempty set, and let $m, n \geq 0$. An (m,n)-*SuperHyperStructure* of arity s is defined by selecting a mapping

$$\odot^{(m,n)} :\ \big(\mathcal{P}^m(S)\big)^s \;\longrightarrow\; \mathcal{P}^n(S).$$

Special cases include:

- $m = n = 0$: ordinary s-ary operations on S,
- $m = 0,\ n = 1$: hyperoperations,
- $s = 1$: superhyperoperations,

Example 1.15 (Everyday (m,n)-SuperHyperStructure: Group itinerary synthesis $(m,n) = (1,2)$)**.** Let S be the airport set above. Define

$$\odot^{(1,2)} :\ \big(\mathcal{P}(S)\big)^2 \longrightarrow \mathcal{P}^2(S)$$

by, for $X, Y \subseteq S$ (nonempty),

$$\odot^{(1,2)}(X,Y) \ :=\ \Big\{ \{h\} \subseteq S \ \Big|\ \exists\, a \in X,\ \exists\, b \in Y :\ h \in a \circ b \Big\}.$$

Thus $\odot^{(1,2)}$ maps *sets of origins/destinations* to a *family of single-hub options*. With $X = \{\mathrm{HND}\}$ and $Y = \{\mathrm{LAX}, \mathrm{JFK}\}$ we obtain, using the computed hubs,

$$\odot^{(1,2)}(X,Y) = \big\{\{\mathrm{ICN}\},\ \{\mathrm{SIN}\},\ \{\mathrm{DOH}\}\big\} \cup \big\{\{\mathrm{ICN}\},\ \{\mathrm{DOH}\}\big\} = \big\{\{\mathrm{ICN}\}, \{\mathrm{SIN}\}, \{\mathrm{DOH}\}\big\} \ \in\ \mathcal{P}^2(S).$$

Hence $\big(S, \odot^{(1,2)}\big)$ is a concrete $(m,n) = (1,2)$-SuperHyperStructure: inputs live in $\mathcal{P}^1(S)$ (sets of endpoints) and outputs in $\mathcal{P}^2(S)$ (families of hub-sets).

Example 1.16 (Combinatorial ester library under multiple catalysts $(m,n) = (1,2),\ s = 2$)**.** Let S be the set of labeled species (acids, alcohols, esters, water, catalysts). Define the finite condition set $C = \{\mathrm{H_2SO_4},\ \text{p-TsOH}\}$. Take

$$A = \{\mathrm{AcOH},\ \mathrm{BzOH}\} \subseteq S, \qquad B = \{\mathrm{EtOH},\ \mathrm{iPrOH}\} \subseteq S,$$

and write the idealized esterification pattern $\text{acid} + \text{alcohol} \ \rightleftharpoons\ \text{ester} + \mathrm{H_2O}$. Define a superhyperoperation

$$\odot^{(1,2)} : \big(\mathcal{P}(S)\big)^2 \longrightarrow \mathcal{P}^2(S)$$

by grouping products *by catalyst*: for each $c \in C$, set

$$Y_c \ :=\ \big\{\ \mathrm{EtOAc},\ \mathrm{iPrOAc},\ \mathrm{EtOBz},\ \mathrm{iPrOBz},\ \mathrm{H_2O}\ \big\} \ \subseteq S,$$

and output

$$\odot^{(1,2)}(A,B) \ :=\ \{\, Y_{\mathrm{H_2SO_4}},\ Y_{\text{p-TsOH}}\, \} \ \in\ \mathcal{P}^2(S).$$

Here the input lives in $\mathcal{P}^1(S)$ (sets of reagents), while the output is a *family* (indexed by C) of product sets, hence level $n = 2$. The number of distinct ester skeletons produced is $|A| \cdot |B| = 2 \cdot 2 = 4$, each appearing in every condition set Y_c.

Example 1.17 (Stoichiometry-windowed polyamide batches from recipe families $(m,n) = (2,1),\ s = 1$)**.** Let S contain species relevant to interfacial polycondensation of hexamethylenediamine (HMD) with sebacoyl chloride ($\mathrm{SebCl_2}$):

$$S = \{\mathrm{HMD},\ \mathrm{SebCl_2},\ \mathrm{PA},\ \mathrm{HCl},\ \text{solvent},\ \text{base}\}.$$

An element of $\mathcal{P}^2(S)$ will encode a *family of recipe sets*. Consider two candidate families

$$\begin{aligned} Y_1 = \big\{\ &\{\mathrm{HMD@1.02\ eq},\ \mathrm{SebCl_2@1.00\ eq},\ \text{solvent},\ \text{base}\},\\ &\{\mathrm{HMD@0.98\ eq},\ \mathrm{SebCl_2@1.00\ eq},\ \text{solvent},\ \text{base}\}\ \big\},\\ Y_2 = \big\{\ &\{\mathrm{HMD@1.20\ eq},\ \mathrm{SebCl_2@1.00\ eq}\},\ \{\mathrm{HMD@0.85\ eq},\ \mathrm{SebCl_2@1.00\ eq}\}\ \big\}, \end{aligned}$$

and let the input be $X = \{Y_1, Y_2\} \in \mathcal{P}^2(S)$. Fix a feasible stoichiometry window for diamine:diacid chloride of $[0.98, 1.02]$ (equivalents). Define

$$\odot^{(2,1)} :\ \mathcal{P}^2(S) \longrightarrow \mathcal{P}(S)$$

by selecting *all* reaction outcomes from atoms (individual recipes) whose equivalence ratio falls in the window and projecting to species:

$$\odot^{(2,1)}(X) \ :=\ \big\{\ \mathrm{PA},\ \mathrm{HCl}\ \big\} \ \subseteq S,$$

since both atoms in Y_1 satisfy the $0.98 - 1.02$ constraint and yield PA + HCl, whereas the atoms in Y_2 are rejected. Thus the input is level $m = 2$ (family of recipe sets) and the output is level $n = 1$ (a set of feasible product species). If one tracks batch identity, one may refine the output to $\{\mathrm{PA}_{(1.02)}, \mathrm{PA}_{(0.98)}, \mathrm{HCl}\}$.

2 Main Results

As the principal outcome of this paper, we investigate four new concepts constructed using the frameworks of HyperStructure and SuperHyperStructure.

2.1 Medical HyperStructure

Medical HyperStructure models patient states and interventions with hyperoperations mapping inputs to feasible outcome sets under guidelines, constraints, valuations.

Definition 2.1 (Medical HyperStructure)**.** Let S_{med} be a nonempty set of *clinically distinguishable patient states*. Let Σ_{med} be a signature of *clinical actions* (diagnostic/therapeutic/triage) with arity map $\mathrm{ar} : \Sigma_{\mathrm{med}} \to \mathbb{Z}_{\geq 1}$. Let $(V, +, \leq)$ be a commutative, partially ordered monoid encoding clinical quantities (e.g. risk, cost, dose, utility), and let

$$\nu : S_{\mathrm{med}} \longrightarrow V$$

be a *clinical valuation*. Let $\mathrm{Obs} : S_{\mathrm{med}} \to \mathcal{O}$ be an *observation map* (biomarkers, vitals, labs). For each $\sigma \in \Sigma_{\mathrm{med}}$ fix

- a *feasibility domain* $D_\sigma \subseteq S_{\mathrm{med}}^{\mathrm{ar}(\sigma)}$,
- a *guideline correspondence* $\mathrm{Feas}_\sigma : S_{\mathrm{med}}^{\mathrm{ar}(\sigma)} \to \mathcal{P}(S_{\mathrm{med}})$,
- a closed convex cone $C_\sigma \subseteq V$ describing admissible net changes (budget/safety).

A *Medical HyperStructure* is a tuple

$$\mathbf{MHS} \;=\; \Big(S_{\mathrm{med}},\ \Sigma_{\mathrm{med}},\ \{\sigma^S\}_{\sigma\in\Sigma_{\mathrm{med}}},\ \nu,\ \mathrm{Obs},\ \{\mathrm{Feas}_\sigma\},\ \{C_\sigma\},\ \{D_\sigma\}\Big),$$

where each *interaction hyperoperation*

$$\sigma^S :\ S_{\mathrm{med}}^{\mathrm{ar}(\sigma)} \longrightarrow \mathcal{P}(S_{\mathrm{med}})$$

satisfies, for all $\vec{s} = (s_1, \ldots, s_{\mathrm{ar}(\sigma)})$:

(MH1) **Feasible nonemptiness:** if $\vec{s} \in D_\sigma$ then $\sigma^S(\vec{s}) \neq \emptyset$.

(MH2) **Guideline compatibility:** $\sigma^S(\vec{s}) \ \subseteq\ \mathrm{Feas}_\sigma(\vec{s})$.

(MH3) **Clinical admissibility (budget/safety):** for every $u \in \sigma^S(\vec{s})$,

$$\nu(u) \ \in\ \nu(s_1) + \cdots + \nu(s_{\mathrm{ar}(\sigma)}) \ +\ C_\sigma.$$

(Taking $C_\sigma = \{0\}$ enforces exact conservation; $C_\sigma \neq \{0\}$ allows bounded change.)

(MH4) **Observational coherence:** there exists an aggregator $H_\sigma : \mathcal{O}^{\,\mathrm{ar}(\sigma)} \to \mathcal{P}(\mathcal{O})$ such that

$$\mathrm{Obs}\big(\sigma^S(\vec{s})\big) \ \subseteq\ H_\sigma\big(\mathrm{Obs}(s_1), \ldots, \mathrm{Obs}(s_{\mathrm{ar}(\sigma)})\big),$$

where $\mathrm{Obs}(A) := \{\mathrm{Obs}(a) \mid a \in A\}$ for $A \subseteq S_{\mathrm{med}}$.

When one forgets $\nu, \mathrm{Obs}, \mathrm{Feas}_\sigma, C_\sigma, D_\sigma$, the pair $(S_{\mathrm{med}}, \{\sigma^S\})$ is a plain hyperstructure in the usual sense.

Example 2.2 (Empiric antibiotics for community–acquired pneumonia (CAP))**.** (cf. [20]) Let S_{med} be the set of patient states $s = (\mathrm{dx}, A, \mathrm{sev}, r)$ consisting of diagnosis label dx, allergy set $A \subseteq \{\beta\text{-lactam}, \text{ macrolide}, \dots\}$, severity class $\mathrm{sev} \in \{1, 2, 3\}$, and current regimen r (possibly $r = \varnothing$). Let F be a hospital formulary and for $r \in F$ write $\mathrm{cost}(r) \geq 0$ and $\mathrm{risk}_{\mathrm{AE}}(r) \geq 0$ (e.g. a calibrated adverse–event score).

Set the valuation space $V = \mathbb{R}^2_{\geq 0}$ with componentwise addition and order, and define

$$\nu(s) \; := \; \big(\mathrm{cost}(r), \; \mathrm{risk}_{\mathrm{AE}}(r)\big).$$

Let Σ_{med} contain a unary interaction $\sigma = \mathrm{abx\_emp}$ (empiric antibiotics). Define the feasibility domain

$$D_\sigma \; := \; \{\, s \in S_{\mathrm{med}} \mid \mathrm{dx} = \mathrm{CAP} \,\},$$

and the policy correspondence $\mathrm{Feas}_\sigma(s)$ as the set of regimens $r' \in F$ that (i) cover guideline pathogens for sev, and (ii) avoid allergies in A. Fix $C_\sigma = [0, C_{\max}] \times [0, R_{\max}]$ (bounding incremental cost and AE risk).

Define the hyperoperation $\sigma^S : S_{\mathrm{med}} \to \mathcal{P}(S_{\mathrm{med}})$ by

$$\sigma^S(s) \; := \; \Big\{ (\mathrm{dx}, A, \mathrm{sev}, r') \; \Big| \; r' \in \mathrm{Feas}_\sigma(s) \Big\}.$$

Then (MH1) holds by guideline nonemptiness, (MH2) by construction, and for any $u \in \sigma^S(s)$,

$$\nu(u) - \nu(s) \; \in \; C_\sigma,$$

which is (MH3). For observational coherence (MH4), if $\mathrm{Obs}(s)$ returns (e.g.) initial vitals/labs and microbiology (possibly pending), set $H_\sigma(o)$ to contain updated summaries after empiric therapy initiation, and observe $\mathrm{Obs}\big(\sigma^S(s)\big) \subseteq H_\sigma(\mathrm{Obs}(s))$.

Example 2.3 (Inpatient insulin titration for hyperglycemia)**.** (cf. [21, 22]) Let S_{med} consist of $s = (G, \; d, \; c)$ with current capillary glucose $G \in [0, \infty)$, basal insulin dose $d \in [0, \infty)$, and clinical context c (e.g. steroid use, NPO status). Let $V = \mathbb{R}^2_{\geq 0}$ with

$$\nu(s) \; := \; \big(\mathrm{hypo\_risk}(s), \; \mathrm{workload}(s)\big),$$

where $\mathrm{hypo\_risk}$ is a surrogate risk (lower is better but we store it as a nonnegative load), and $\mathrm{workload}$ encodes nursing intensity.

Let $\sigma = \mathrm{titrate}$ be unary. The feasibility domain is

$$D_\sigma = \{\, (G, d, c) \mid \text{no active contraindication to insulin adjustment in } c \,\}.$$

Fix step bounds $\Delta^-, \Delta^+ \geq 0$, and define the policy correspondence

$$\mathrm{Feas}_\sigma(G, d, c) \; := \; \{\, d' \in [\, \max(0, d - \Delta^-), \; d + \Delta^+] \; | \; \text{unit protocol for } c \,\}.$$

Let $C_\sigma = [0, R_{\max}] \times [0, W_{\max}]$. Define

$$\sigma^S(G, d, c) \; := \; \Big\{ (G', d', c) \; \Big| \; d' \in \mathrm{Feas}_\sigma(G, d, c), \; G' \in \mathrm{Pred}(G, d, d', c) \Big\},$$

where Pred is a (set–valued) next–glucose predictor consistent with the protocol (e.g. bracketing due to meal timing). Then (MH1)–(MH2) are immediate. For (MH3), protocol design yields $\nu(G', d', c) - \nu(G, d, c) \in C_\sigma$ (bounded changes in risk/workload). For (MH4), taking $\mathrm{Obs}(s) = (G, c)$ and $H_\sigma(G, c)$ the set of admissible next measurements for protocol steps ensures $\mathrm{Obs}\big(\sigma^S(s)\big) \subseteq H_\sigma(\mathrm{Obs}(s))$.

Example 2.4 (Neuroimaging triage for acute head injury)**.** (cf. [23]) Let S_{med} be states

$$s = (\mathrm{GCS}, \; \mathrm{on\_OAC}, \; \mathrm{eGFR}, \; \mathrm{imaging})$$

with Glasgow Coma Scale $\mathrm{GCS} \in \{3, \dots, 15\}$, oral anticoagulant flag $\mathrm{on\_OAC} \in \{0, 1\}$, renal function $\mathrm{eGFR} \in [0, \infty)$, and current imaging plan

$$\mathrm{imaging} \in \{\varnothing, \mathrm{CT\_NC}, \mathrm{CT\_C}, \mathrm{MRI}\}$$

. Let $V = \mathbb{R}^2_{\geq 0}$ with

$$\nu(s) \;:=\; \big(\mathrm{time\_to\_result}(s),\ \mathrm{radiation\_dose}(s)\big).$$

Let $\sigma = \mathrm{neuro\_triage}$ (unary). The feasibility domain requires standard trauma criteria (e.g. loss of consciousness or focal deficit ⇒ eligible). The policy correspondence encodes contraindications (e.g. if $\mathrm{eGFR} < 30$ then CT_C disallowed; MRI access/time limits).

Set $C_\sigma = [0, T_{\max}] \times [0, D_{\max}]$. Define

$$\sigma^S(s) \;:=\; \Big\{ (\mathrm{GCS}, \mathrm{on\_OAC}, \mathrm{eGFR}, \mathrm{plan}) \;\Big|\; \mathrm{plan} \in \mathrm{Feas}_\sigma(s) \Big\}.$$

Then (MH1) nonemptiness follows from at least one admissible modality; (MH2) by construction; (MH3) holds since ν changes (time, dose) are bounded by C_σ under protocolized pathways; (MH4) with $\mathrm{Obs}(s)$ the presenting exam and FAST score, and H_σ the guideline–consistent post–triage summaries, one has $\mathrm{Obs}(\sigma^S(s)) \subseteq H_\sigma(\mathrm{Obs}(s))$.

Example 2.5 (Anticoagulation initiation in atrial fibrillation)**.** (cf. [24, 25]) Let S_{med} be states

$$s = (\mathrm{CHA2DS2\text{-}VASc},\ \mathrm{HAS\text{-}BLED},\ \mathrm{eGFR},\ \mathrm{agent})$$

, with risk scores and current agent

$$\mathrm{agent} \in \{\varnothing,\ \mathrm{DOAC\_low},\ \mathrm{DOAC\_std},\ \mathrm{warfarin}\}$$

. Let $V = \mathbb{R}^2_{\geq 0}$ and

$$\nu(s) \;:=\; \big(\mathrm{stroke\_risk\_est}(s),\ \mathrm{bleed\_risk\_est}(s)\big).$$

Let $\sigma = \mathrm{anticoag\_start}$ (unary) with feasibility domain excluding absolute contraindications (e.g. active bleed). The policy correspondence Feas_σ selects agents/doses admissible for eGFR and risk strata per guideline. Take $C_\sigma = [-S_{\max}, B_{\max}] \times [0, B_{\max}]$ (where negative first component models a *decrease* in stroke risk within allowed range, and the second bounds bleeding–risk increase). Define

$$\sigma^S(s) \;:=\; \Big\{ (\mathrm{CHA2DS2\text{-}VASc},\ \mathrm{HAS\text{-}BLED},\ \mathrm{eGFR},\ \mathrm{agent}') \;\Big|\; \mathrm{agent}' \in \mathrm{Feas}_\sigma(s) \Big\}.$$

Then (MH1)–(MH2) hold by construction. For (MH3), any $u \in \sigma^S(s)$ satisfies $\nu(u) - \nu(s) \in C_\sigma$ (bounded improvement in stroke risk and bounded change in bleeding risk). For (MH4), let $\mathrm{Obs}(s)$ contain labs (INR, creatinine) and H_σ collect the post–initiation monitoring targets; then $\mathrm{Obs}\big(\sigma^S(s)\big) \subseteq H_\sigma(\mathrm{Obs}(s))$.

Theorem 2.6 (Medical HyperStructure is a HyperStructure)**.** *Let* **MHS** *be as in Definition 2.1. Then the* underlying family of maps

$$\mathcal{H}_{\mathrm{med}} \;:=\; \big(S_{\mathrm{med}},\ \Sigma_{\mathrm{med}},\ \{\sigma^S\}_{\sigma \in \Sigma_{\mathrm{med}}}\big)$$

is a (baseline) HyperStructure on S_{med}.

Proof. Fix $\sigma \in \Sigma_{\mathrm{med}}$ and abbreviate $k = \mathrm{ar}(\sigma)$. By Definition 2.1, for every input k-tuple $\vec{s} \in S^k_{\mathrm{med}}$ the map

$$\sigma^S : S^k_{\mathrm{med}} \longrightarrow \mathcal{P}(S_{\mathrm{med}})$$

is *well-typed*, i.e., $\sigma^S(\vec{s}) \in \mathcal{P}(S_{\mathrm{med}})$. Hence σ^S is a set-valued operation on S_{med}.

Since this holds for every $\sigma \in \Sigma_{\mathrm{med}}$, the collection $\{\sigma^S\}_{\sigma \in \Sigma_{\mathrm{med}}}$ satisfies precisely the data required in the baseline definition of a HyperStructure: a family of maps $S^{\mathrm{ar}(\sigma)}_{\mathrm{med}} \to \mathcal{P}(S_{\mathrm{med}})$. Therefore $\mathcal{H}_{\mathrm{med}}$ is a HyperStructure on S_{med}. □

2.2 Medical SuperHyperStructure

Medical SuperHyperStructure extends to iterated powersets, capturing cohorts, multi-stage care pathways, and policies with lifted feasibility, observational coherence, bounded resources.

Definition 2.7 (Lifts to iterated powersets)**.** For $k \geq 0$, define $\mathcal{P}^0(()S_{\mathrm{med}}) = S_{\mathrm{med}}$ and $\mathcal{P}^{k+1}(()S_{\mathrm{med}}) = \mathcal{P}(\mathcal{P}^k(()S_{\mathrm{med}}))$. For $Z \in \mathcal{P}^k(()S_{\mathrm{med}})$, its set of *atoms* is defined recursively by $\mathrm{At}(z) = \{z\}$ when $k = 0$ and $\mathrm{At}(Z) = \bigcup_{z \in Z} \mathrm{At}(z)$ for $k \geq 1$. The observation map lifts as $\mathrm{Obs}^{\uparrow 0} = \mathrm{Obs}$ and $\mathrm{Obs}^{\uparrow(k+1)}(X) = \{\mathrm{Obs}^{\uparrow k}(x) \mid x \in X\}$.

Definition 2.8 (Medical SuperHyperStructure of order (m,n))**.** Fix integers $m, n \geq 0$ and a signature Σ_{med} with arities $\mathrm{ar}(\sigma)$. For each $\sigma \in \Sigma_{\mathrm{med}}$ choose

$$D_\sigma^{(m)} \subseteq \big(\mathcal{P}^m(()S_{\mathrm{med}})\big)^{\mathrm{ar}(\sigma)}, \qquad \mathrm{Feas}_\sigma^{(m,n)} : \big(\mathcal{P}^m(()S_{\mathrm{med}})\big)^{\mathrm{ar}(\sigma)} \longrightarrow \mathcal{P}\big(\mathcal{P}^n(()S_{\mathrm{med}})\big),$$

and a closed convex cone $C_\sigma \subseteq V$ in the valuation space $(V, +, \leq)$ with $\nu : S_{\mathrm{med}} \to V$ as in Definition 2.1. A *Medical SuperHyperStructure of order* (m,n) is a tuple

$$\mathbf{MSHS} = \Big(S_{\mathrm{med}},\, \Sigma_{\mathrm{med}},\, \{\odot_\sigma^{(m,n)}\}_{\sigma \in \Sigma_{\mathrm{med}}},\, \nu,\, \mathrm{Obs},\, \{\mathrm{Feas}_\sigma^{(m,n)}\},\, \{C_\sigma\},\, \{D_\sigma^{(m)}\}\Big),$$

where each *superhyperoperation*

$$\odot_\sigma^{(m,n)} : \big(\mathcal{P}^m(()S_{\mathrm{med}})\big)^{\mathrm{ar}(\sigma)} \longrightarrow \mathcal{P}^n(()S_{\mathrm{med}})$$

satisfies, for all inputs $\vec{X} = (X_1, \ldots, X_{\mathrm{ar}(\sigma)})$:

(MSH1) **Feasible nonemptiness:** if $\vec{X} \in D_\sigma^{(m)}$ then $\odot_\sigma^{(m,n)}(\vec{X}) \neq \emptyset$.

(MSH2) **Guideline compatibility (lifted):**

$$\odot_\sigma^{(m,n)}(\vec{X}) \subseteq \mathrm{Feas}_\sigma^{(m,n)}(\vec{X}).$$

(MSH3) **Atomic clinical admissibility:** for every $Y \in \odot_\sigma^{(m,n)}(\vec{X})$ and every $u \in \mathrm{At}(Y)$ there exist $s_i \in \mathrm{At}(X_i)$ such that

$$\nu(u) \in \nu(s_1) + \cdots + \nu(s_{\mathrm{ar}(\sigma)}) + C_\sigma.$$

(MSH4) **Observational coherence (lifted):** there exists

$$H_\sigma^{(m,n)} : \big(\mathcal{P}^m(()\mathcal{O})\big)^{\mathrm{ar}(\sigma)} \longrightarrow \mathcal{P}^n(()\mathcal{O})$$

with

$$\mathrm{Obs}^{\uparrow n}\big(\odot_\sigma^{(m,n)}(\vec{X})\big) \subseteq H_\sigma^{(m,n)}\big(\mathrm{Obs}^{\uparrow m}(X_1), \ldots, \mathrm{Obs}^{\uparrow m}(X_{\mathrm{ar}(\sigma)})\big).$$

When $(m,n) = (0,1)$ and $\odot_\sigma^{(0,1)} = \sigma^S$, Definition 2.8 reduces to the Medical HyperStructure in Definition 2.1.

Example 2.9 (Sepsis order–set generation with guideline tiers: $(m,n) = (1,2)$)**.** Let S_{med} consist of patient states $s = (\mathrm{dx}, \mathrm{sev}, A, \mathrm{labs}, r)$ with diagnosis dx, severity class sev, allergy set A, basic labs labs, and current regimen r. Let $V = \mathbb{R}^2_{\geq 0}$ with valuation $\nu(s) = (\mathrm{cost}(r),\ \mathrm{AE\_risk}(r))$. Fix an arity-1 symbol $\sigma = \mathrm{sepsis\_orders}$. For $X \in \mathcal{P}^1(()S_{\mathrm{med}}) = \mathcal{P}(S_{\mathrm{med}}) \setminus \{\emptyset\}$, define

$$\odot_\sigma^{(1,2)}(X) := \Big\{ \mathcal{Y} \subseteq S_{\mathrm{med}} \;\Big|\; \exists \text{ finite tier set } \mathcal{T} \text{ s.t. } \mathcal{Y} = \big\{ u \in \mathrm{Feas}_\sigma^{(1,2)}(X) \;\big|\; u \in \textstyle\bigcup_{s \in X} \bigcup_{t \in \mathcal{T}} \sigma_t^S(s) \big\} \Big\}.$$

Here each σ_t^S is a single-tier order–set rule (e.g. "severe sepsis, β-lactam allergy"), and $\mathrm{Feas}_\sigma^{(1,2)}(X)$ encodes policy filters (e.g. formulary, renal dosing). *Checks.* If every $s \in X$ meets screening criteria $D_\sigma^{(1)}$, the union over finitely many tiers is nonempty (MSH1). By construction $\odot_\sigma^{(1,2)}(X) \subseteq \mathrm{Feas}_\sigma^{(1,2)}(X)$ (MSH2). For any $Y \in \odot_\sigma^{(1,2)}(X)$ and $u \in \mathrm{At}(Y)$, there is $s \in \mathrm{At}(X)$ with $u \in \sigma_t^S(s)$, hence $\nu(u) \in \nu(s) + C_\sigma$ for a cone $C_\sigma = [0, C_{\max}] \times [0, R_{\max}]$ (MSH3). With $\mathrm{Obs}(s)$ the presenting vitals/labs and $H_\sigma^{(1,2)}$ collecting the post–order observable summaries at level-2, we obtain $\mathrm{Obs}^{\uparrow 2}(\odot_\sigma^{(1,2)}(X)) \subseteq H_\sigma^{(1,2)}(\mathrm{Obs}^{\uparrow 1}(X))$ (MSH4).

Example 2.10 (Multidisciplinary oncology consensus from proposal families: $(m, n) = (2, 1)$)**.** Let $S_{\rm med}$ encode single–patient next states $s = (\text{plan}, \text{tox}, \text{cost})$ with $\text{plan} \in \{\text{Surg},\ \text{Chemo},\ \text{RT},\ \text{Combo}\}$, and let $V = \mathbb{R}^2_{\geq 0}$ with $\nu(s) = (\text{tox}, \text{cost})$. One patient is under discussion; input $X \in \mathcal{P}^2(()S_{\rm med})$ is a *family of proposal sets*, e.g. one set per specialty (surgical, medical, radiation). Define the arity-1 operation $\sigma = \text{tumorboard}$ by

$$\odot_\sigma^{(2,1)}(X) \ := \ \Big\{ u \in S_{\rm med} \ \Big| \ \exists Y \in X,\ \exists s \in Y \text{ with } u \in \Pi(s) \ \text{ and } \ \nu(u) - \nu(s) \in C_\sigma \Big\},$$

where Π is a reconciliation map (e.g. dose-adjusted variants, schedule alignments) and $C_\sigma = \{(t, c) \mid 0 \leq t \leq T_{\max},\ 0 \leq c \leq C_{\max}\}$ bounds admissible changes. *Checks.* If X contains at least one feasible proposal set $Y \in D_\sigma^{(2)}$, then $\odot_\sigma^{(2,1)}(X) \neq \emptyset$ (MSH1). Guideline filters are enforced by $u \in \text{Feas}_\sigma^{(2,1)}(X)$ (MSH2). For $u \in \odot_\sigma^{(2,1)}(X)$ we have u derived from some $s \in \text{At}(Y)$ for $Y \in X$, giving $\nu(u) \in \nu(s) + C_\sigma$ (MSH3). Let $\text{Obs}(s)$ collect key endpoints; $H_\sigma^{(2,1)}$ compresses the family–level observations to a single plan summary, hence (MSH4).

Example 2.11 (Chronic disease panel selection as bundle–of–bundles: $(m, n) = (0, 2)$)**.** Let $S_{\rm med}$ be patient states $s = (\text{dx}, \text{stage}, \text{comorb})$ for a chronic condition. We generate *monitoring bundles* (labs, imaging, visits) under multiple expert templates. Let $V = \mathbb{R}^2_{\geq 0}$ with $\nu(s) = (\text{burden}(s),\ \text{cost}(s))$ computed from the bundle attached to s. For arity-1 symbol $\sigma = \text{monitoring}$ and $s \in S_{\rm med}$, define

$$\odot_\sigma^{(0,2)}(s) \ := \ \Big\{ \mathcal{Y} \subseteq S_{\rm med} \ \Big| \ \exists \text{ finite template set } \mathcal{T} \text{ s.t. } \mathcal{Y} = \big\{ \tau(s) \ \big| \ \tau \in \mathcal{T} \big\} \Big\}.$$

Here each τ attaches a monitoring bundle consistent with stage/comorbidity and policy. *Checks.* Nonemptiness holds if $s \in D_\sigma^{(0)}$ (screened) (MSH1), and $\mathcal{Y} \subseteq \text{Feas}_\sigma^{(0,2)}(s)$ (MSH2). For any $Y \in \odot_\sigma^{(0,2)}(s)$ and $u \in \text{At}(Y)$, $\nu(u) - \nu(s) \in C_\sigma$ with $C_\sigma = [0, B_{\max}] \times [0, C_{\max}]$ bounding burden/cost (MSH3). Taking $\text{Obs}(s)$ as current schedule and $H_\sigma^{(0,2)}$ the level-2 collection of admissible next schedules implies (MSH4).

Example 2.12 (Renal–function–robust anticoagulation dosing: $(m, n) = (1, 1)$)**.** Consider $S_{\rm med}$ with states $s = (\text{eGFR}, \text{agent}, \text{dose})$. Let $V = \mathbb{R}^2_{\geq 0}$ with $\nu(s) = (\text{stroke\_risk}(s),\ \text{bleed\_risk}(s))$. The input $X \in \mathcal{P}^1(()S_{\rm med})$ collects admissible *renal scenarios* (e.g. eGFR realizations from measurement uncertainty or near-term trend). For the arity-1 symbol $\sigma = \text{anticoag\_robust}$, set

$$\odot_\sigma^{(1,1)}(X) \ := \ \Big\{ u \in S_{\rm med} \ \Big| \ \exists s \in X \ \text{ with } u \in \text{Feas}_\sigma^{(1,1)}(X) \cap \sigma^S(s),\ \nu(u) - \nu(s) \in C_\sigma \Big\},$$

where $\sigma^S(s)$ returns dose adjustments admissible for the specific eGFR and C_σ bounds the net change in risks. Then (MSH1)–(MSH2) hold by feasibility/policy; (MSH3) holds with the chosen C_σ; (MSH4) follows by lifting lab/INR observations through $H_\sigma^{(1,1)}$ from set–level inputs to set–level outputs.

Theorem 2.13 (Medical SuperHyperStructure is a SuperHyperStructure)**.** *Let* MSHS *be as above. Then the* underlying family of maps

$$\mathcal{SH}_{\rm med}^{(m,n)} \ := \ \Big(S_{\rm med},\ \Sigma_{\rm med},\ \{\odot_\sigma^{(m,n)}\}_{\sigma \in \Sigma_{\rm med}} \Big)$$

is an (m, n)*-SuperHyperStructure on* $S_{\rm med}$*.*

Proof. Fix $\sigma \in \Sigma_{\rm med}$ and abbreviate $k = \text{ar}(\sigma)$. By definition,

$$\odot_\sigma^{(m,n)} : \ \big(\mathcal{P}^m(S_{\rm med})\big)^k \longrightarrow \mathcal{P}^n(S_{\rm med})$$

is *well-typed*. Thus for every input $\vec{X} \in \big(\mathcal{P}^m(S_{\rm med})\big)^k$, its image $\odot_\sigma^{(m,n)}(\vec{X})$ lies in $\mathcal{P}^n(S_{\rm med})$. Since this holds for all σ, the collection $\{\odot_\sigma^{(m,n)}\}$ satisfies exactly the typing required in the baseline definition of an (m, n)-SuperHyperStructure. Therefore $\mathcal{SH}_{\rm med}^{(m,n)}$ is an (m, n)-SuperHyperStructure on $S_{\rm med}$. ☐

2.3 Healthcare HyperStructure

Healthcare coordinates prevention, diagnosis, treatment, and rehabilitation services, balancing patient outcomes, access, safety, cost, and workforce sustainability across populations equitably (cf. [26,26–29]). Healthcare HyperStructure models system states with hyperoperations mapping actions to sets of feasible next states under policy and KPI constraints.

Definition 2.14 (Healthcare HyperStructure)**.** Let S_{hcs} be a nonempty set of *healthcare system states* (e.g., tuples collecting patient cohorts, staff rosters, bed/capacity occupancy, queues, inventories). Let Σ_{hcs} be a signature of *operational actions* (triage, admit, transfer, schedule, allocate, discharge, bill, etc.) with arity map $\mathrm{ar} : \Sigma_{\mathrm{hcs}} \to \mathbb{Z}_{\geq 1}$.

Let $(V, +, \leq)$ be a commutative, partially ordered monoid encoding multi-criteria key performance indicators (KPI) such as *cost*, *workload*, *risk*, *time*, *utility*. Let

$$\nu : S_{\mathrm{hcs}} \longrightarrow V$$

be a *system valuation* (state $\mapsto$ KPI vector). Let $\mathrm{Obs} : S_{\mathrm{hcs}} \to \mathcal{O}$ be an *observation map* returning observable summaries (e.g., EHR-derived widgets: census, wait times, backlog).

For each $\sigma \in \Sigma_{\mathrm{hcs}}$ fix:

- a *feasibility domain* $D_\sigma \subseteq S_{\mathrm{hcs}}^{\mathrm{ar}(\sigma)}$ (policy, regulatory, and physical constraints),
- a *policy correspondence* $\mathrm{Feas}_\sigma : S_{\mathrm{hcs}}^{\mathrm{ar}(\sigma)} \to \mathcal{P}(S_{\mathrm{hcs}})$ mapping inputs to allowed next states,
- a closed convex cone $C_\sigma \subseteq V$ describing admissible net KPI changes (budget/safety/capacity margins).

A *Healthcare HyperStructure* is a tuple

$$\mathbf{HHS} \;=\; \Big(S_{\mathrm{hcs}},\ \Sigma_{\mathrm{hcs}},\ \{\sigma^S\}_{\sigma \in \Sigma_{\mathrm{hcs}}},\ \nu,\ \mathrm{Obs},\ \{\mathrm{Feas}_\sigma\},\ \{C_\sigma\},\ \{D_\sigma\}\Big),$$

where each *interaction hyperoperation*

$$\sigma^S :\ S_{\mathrm{hcs}}^{\mathrm{ar}(\sigma)} \longrightarrow \mathcal{P}(S_{\mathrm{hcs}})$$

satisfies, for all $\vec{s} = (s_1, \ldots, s_{\mathrm{ar}(\sigma)})$:

(HH1) **Feasible nonemptiness:** if $\vec{s} \in D_\sigma$ then $\sigma^S(\vec{s}) \neq \emptyset$.

(HH2) **Policy compatibility:** $\sigma^S(\vec{s}) \ \subseteq\ \mathrm{Feas}_\sigma(\vec{s})$.

(HH3) **KPI admissibility (budget/safety/capacity):** for every $u \in \sigma^S(\vec{s})$,

$$\nu(u) \ \in\ \nu(s_1) + \cdots + \nu(s_{\mathrm{ar}(\sigma)}) \ +\ C_\sigma.$$

(Choosing $C_\sigma = \{0\}$ enforces exact KPI additivity; $C_\sigma \neq \{0\}$ allows bounded deviations.)

(HH4) **Observational coherence:** there exists an aggregator $H_\sigma : \mathcal{O}^{\,\mathrm{ar}(\sigma)} \to \mathcal{P}(\mathcal{O})$ with

$$\mathrm{Obs}\big(\sigma^S(\vec{s})\big) \ \subseteq\ H_\sigma\big(\mathrm{Obs}(s_1), \ldots, \mathrm{Obs}(s_{\mathrm{ar}(\sigma)})\big),$$

where $\mathrm{Obs}(A) := \{\mathrm{Obs}(a) \mid a \in A\}$ for $A \subseteq S_{\mathrm{hcs}}$.

For binary σ the induced operation on $\mathcal{P}(S_{\mathrm{hcs}})$ is $A \ \sigma\ B := \bigcup_{a \in A,\, b \in B} \sigma^S(a, b)$, as standard in hyperalgebra.

Example 2.15 (ED triage and admission policy (arity 1))**.** (cf. [30, 31]) Let S_{hcs} collect ED system states $s = (q,\ b,\ r)$ where $q = (q_1, \ldots, q_L) \in \mathbb{N}^L$ are queue counts by acuity tier, $b \in \mathbb{N}$ is the number of free inpatient beds, and $r \in \mathbb{R}^d_{\geq 0}$ aggregates available resources (nurses, bays). Let $V = \mathbb{R}^2_{\geq 0}$ with componentwise $+$ and $\leq$, and set

$$\nu(s) \ :=\ \big(\mathrm{wait\_proxy}(s),\ \mathrm{overtime\_proxy}(s)\big).$$

Let $\sigma = \mathrm{triage\_admit}$ be unary. The feasibility domain D_σ requires (e.g.) r above a minimal safe staffing threshold. Define the policy correspondence

$$\mathrm{Feas}_\sigma(s) \ :=\ \Big\{\, s' = (q', b', r') \ \Big|\ \exists\, u \in \mathbb{N}^L :\ \sum_{\ell=1}^{L} u_\ell \leq b,\ 0 \leq u_\ell \leq q_\ell,$$

$$q' = q - u, \quad b' = b - \textstyle\sum_\ell u_\ell, \quad r' = \varphi(r, u), \text{ and all unit rules hold}\Big\},$$

where φ updates resources (e.g., assigns nurse load). Fix a cone $C_\sigma = \{(x_1, x_2) \in \mathbb{R}^2_{\geq 0} \mid x_1 \leq W_{\max},\ x_2 \leq O_{\max}\}$. Define the hyperoperation

$$\sigma^S(s) \;:=\; \mathrm{Feas}_\sigma(s).$$

Then (HH1) holds whenever $b > 0$ or a fast–track is available; (HH2) is by definition; for any $u \in \sigma^S(s)$ one has $\nu(u) - \nu(s) \in C_\sigma$ (HH3), i.e. KPI changes are bounded; and with $\mathrm{Obs}(s)$ the live dashboard (census, queue, staffing), one can take $H_\sigma(\mathrm{Obs}(s))$ as the set of post–admit dashboards allowed by policy (HH4).

Example 2.16 (Operating room (OR) day scheduling under block policy (arity 1))**.** (cf. [32, 33]) Let S_{hcs} contain states $s = (\mathcal{J},\ B,\ \kappa)$ where $\mathcal{J}$ is a finite set of surgical cases with predicted durations $\{d_j\}_{j\in\mathcal{J}}$, B are block assignments per service, and κ summarizes turnover times and staffing. Let

$$V = \mathbb{R}^2_{\geq 0}, \qquad \nu(s) \;:=\; \big(\mathrm{overtime}(s),\ \mathrm{idle\_time}(s)\big).$$

Let $\sigma = \mathrm{build\_schedule}$ (unary). Feasibility D_σ enforces block ownership, precedence constraints, and room limits. Define

$$\mathrm{Feas}_\sigma(s) \;:=\; \Big\{ s' = (\mathcal{J}, B, \kappa') \;\Big|\; \exists \text{ feasible timetable } \tau :\ \text{rooms} \times \text{time} \to \mathcal{J} \cup \{\varnothing\},$$

$$\text{satisfying blocks } B,\ \text{turnovers under } \kappa,\ \text{and capacity limits; } \kappa' = \psi(\kappa, \tau)\Big\}.$$

Let $C_\sigma = \{(x_1, x_2) \in \mathbb{R}^2_{\geq 0} \mid x_1 \leq O_{\max},\ x_2 \leq I_{\max}\}$. Set

$$\sigma^S(s) \;:=\; \mathrm{Feas}_\sigma(s).$$

Then (HH1) follows from existence of a feasible packing (e.g. first–fit with overtime cap); (HH2) is immediate; (HH3) holds by the KPI bounds; and $\mathrm{Obs}(s)$ (case list, blocks) passes through an aggregator H_σ (Gantt snapshots, KPIs), giving (HH4).

Example 2.17 (Inpatient bed management with inter-unit transfers (arity 1))**.** (cf. [34,35]) Let S_{hcs} be hospital states $s = (\mathbf{c},\ \mathbf{k},\ \iota)$ where $\mathbf{c} = (c_1, \ldots, c_U)$ are current occupancies per unit, $\mathbf{k} = (k_1, \ldots, k_U)$ are capacities, and ι collects isolation/cohorting constraints (matrix of compatibilities). Let $V = \mathbb{R}^2_{\geq 0}$ with

$$\nu(s) \;:=\; \big(\mathrm{boarding\_hours}(s),\ \mathrm{transfer\_load}(s)\big).$$

Let $\sigma = \mathrm{transfer}$ be unary. Feasibility D_σ requires free capacity respecting isolation. Define the policy correspondence as

$$\mathrm{Feas}_\sigma(s) \;:=\; \Big\{ s' = (\mathbf{c}', \mathbf{k}, \iota) \;\Big|\; \exists \text{ flow } f_{uv} \in \mathbb{N}_{\geq 0} \text{ with}$$

$$\mathbf{c}' = \mathbf{c} + \sum_v f_{v\cdot} - \sum_u f_{\cdot u},\ 0 \leq c'_u \leq k_u,\ \text{and cohorting constraints respected}\Big\}.$$

Let $C_\sigma = \{(x_1, x_2) \in \mathbb{R}^2_{\geq 0} \mid x_1 \leq B_{\max},\ x_2 \leq L_{\max}\}$. Set

$$\sigma^S(s) \;:=\; \mathrm{Feas}_\sigma(s).$$

Then (HH1) holds whenever a feasible integer flow exists (classical max–flow with side constraints); (HH2) is by definition; (HH3) holds since boarding and transfer workload changes are bounded; (HH4) uses $\mathrm{Obs}(s)$ (unit census, blocks) and H_σ (bed board after transfers).

Example 2.18 (Pharmacy inventory replenishment with service-level targets (arity 1))**.** (cf. [36, 37]) Let S_{hcs} be pharmacy states $s = (x,\ \hat{d},\ L)$ with current on-hand vector $x \in \mathbb{R}^n_{\geq 0}$, forecasted demand $\hat{d} \in \mathbb{R}^n_{\geq 0}$, and supplier lead times $L \in \mathbb{R}^n_{\geq 0}$. Let

$$V = \mathbb{R}^2_{\geq 0}, \qquad \nu(s) \;:=\; \big(\mathrm{stockout\_risk}(s),\ \mathrm{holding\_cost}(s)\big).$$

Let $\sigma = \mathrm{replenish}$ (unary) with feasibility domain enforcing order windows and budget. Define

$$\mathrm{Feas}_\sigma(s) \;:=\; \Big\{ s' = (x + q,\ \hat{d},\ L) \;\Big|\; q \in \mathcal{Q}(s) \subseteq \mathbb{R}^n_{\geq 0},\ \mathbf{1}^\top c \odot q \leq \mathrm{Budget},\ \text{service levels met}\Big\},$$

where $\mathcal{Q}(s)$ is the admissible order set and c the unit costs. Take $C_\sigma = \{(x_1, x_2) \in \mathbb{R}^2_{\geq 0} \mid x_1 \leq R_{\max},\ x_2 \leq H_{\max}\}$. Then

$$\sigma^S(s) \;:=\; \mathrm{Feas}_\sigma(s)$$

satisfies (HH1)–(HH3) by construction; with $\mathrm{Obs}(s)$ the inventory dashboard and H_σ the post-order KPIs (fill rate curves), (HH4) holds.

Example 2.19 (Outpatient clinic overbooking policy (arity 1))**.** (cf. [38,39]) Let $S_{\rm hcs}$ consist of $s = (\mathcal{A},\ \rho,\ \theta)$ where $\mathcal{A}$ is the current appointment list, $\rho \in [0,1]$ a no-show estimate, and θ resource parameters (rooms, providers). Let $V = \mathbb{R}^2_{\geq 0}$ with

$$\nu(s) \ := \ \big(\text{expected\_wait}(s),\ \text{overtime\_risk}(s)\big).$$

Let $\sigma = \text{overbook}$ be unary. Feasibility D_σ enforces policy bounds (e.g. overbooking cap per hour). Define

$$\text{Feas}_\sigma(s) \ := \ \Big\{\, s' = (\mathcal{A}', \rho, \theta) \ \Big|\ \mathcal{A}' \in \mathcal{G}(\mathcal{A}, \rho, \theta) \text{ (admissible insertions/deferrals), service constraints satisfied} \Big\}.$$

Let $C_\sigma = \{(x_1, x_2) \in \mathbb{R}^2_{\geq 0} \mid x_1 \leq W_{\max},\ x_2 \leq O_{\max}\}$. Set $\sigma^S(s) = \text{Feas}_\sigma(s)$. Then (HH1)–(HH3) hold immediately, and with $\text{Obs}(s)$ the booked template and KPI projections, the aggregator H_σ returns the set of post–policy templates and projections, giving (HH4).

Theorem 2.20 (Healthcare HyperStructure is a HyperStructure)**.** *Let* **HHS** *be a Healthcare HyperStructure as in Definition 2.14, i.e.,*

$$\mathbf{HHS} = \Big(S_{\rm hcs},\ \Sigma_{\rm hcs},\ \{\sigma^S\}_{\sigma \in \Sigma_{\rm hcs}},\ \nu,\ \text{Obs},\ \{\text{Feas}_\sigma\},\ \{C_\sigma\},\ \{D_\sigma\}\Big),$$

where, for each $\sigma \in \Sigma_{\rm hcs}$ with $k = \text{ar}(\sigma)$, $\sigma^S : S^k_{\rm hcs} \to \mathcal{P}(S_{\rm hcs})$ satisfies the axioms (HH1)–(HH4). Then the underlying triple

$$\mathcal{H}_{\rm hcs} \ := \ \big(S_{\rm hcs},\ \Sigma_{\rm hcs},\ \{\sigma^S\}_{\sigma \in \Sigma_{\rm hcs}}\big)$$

is a (baseline) HyperStructure on $S_{\rm hcs}$.

Proof. Fix $\sigma \in \Sigma_{\rm hcs}$ and abbreviate $k = \text{ar}(\sigma)$. By Definition 2.14, the *type* of σ^S is

$$\sigma^S : S^k_{\rm hcs} \longrightarrow \mathcal{P}(S_{\rm hcs}).$$

Concretely, for every input $\vec{s} = (s_1, \ldots, s_k) \in S^k_{\rm hcs}$ the value $\sigma^S(\vec{s})$ is a subset of $S_{\rm hcs}$. Indeed, axiom (HH2) gives the explicit inclusion

$$\sigma^S(\vec{s}) \ \subseteq \ \text{Feas}_\sigma(\vec{s}),$$

and the policy map has codomain $\text{Feas}_\sigma(\vec{s}) \in \mathcal{P}(S_{\rm hcs})$ by definition. Therefore $\sigma^S(\vec{s}) \in \mathcal{P}(S_{\rm hcs})$ for all $\vec{s}$, i.e., σ^S is a well-typed set-valued operation on $S_{\rm hcs}$.

Since this argument applies to *every* $\sigma \in \Sigma_{\rm hcs}$, the family $\{\sigma^S\}_{\sigma \in \Sigma_{\rm hcs}}$ satisfies exactly the data required in the baseline definition of a HyperStructure:

$$\forall \sigma \in \Sigma_{\rm hcs} : \quad \sigma^S : S^{\text{ar}(\sigma)}_{\rm hcs} \longrightarrow \mathcal{P}(S_{\rm hcs}).$$

Hence $\mathcal{H}_{\rm hcs}$ is a HyperStructure on $S_{\rm hcs}$. ☐

2.4 Healthcare SuperHyperStructure

Healthcare SuperHyperStructure extends hyperstructures using iterated powersets, capturing hierarchical, multi-facility decisions with lifted feasibility, observation coherence, and bounded performance changes.

Definition 2.21 (Lifts, atoms, and lifted observations)**.** For $k \geq 0$, define $\mathcal{P}^0(()S_{\rm hcs}) = S_{\rm hcs}$ and $\mathcal{P}^{k+1}(()S_{\rm hcs}) = \mathcal{P}(\mathcal{P}^k(()S_{\rm hcs}))$. For $Z \in \mathcal{P}^k(()S_{\rm hcs})$, its set of *atoms* is defined by $\text{At}(z) = \{z\}$ when $k = 0$ and $\text{At}(Z) = \bigcup_{z \in Z} \text{At}(z)$ for $k \geq 1$. The observation map lifts as $\text{Obs}^{\uparrow 0} = \text{Obs}$ and $\text{Obs}^{\uparrow(k+1)}(X) = \{\text{Obs}^{\uparrow k}(x) \mid x \in X\}$.

Definition 2.22 (Healthcare SuperHyperStructure of order (m, n))**.** Fix integers $m, n \geq 0$ and a signature $\Sigma_{\rm hcs}$ with arities $\text{ar}(\sigma)$. For each $\sigma \in \Sigma_{\rm hcs}$ choose

$$D^{(m)}_\sigma \ \subseteq \ \big(\mathcal{P}^m(()S_{\rm hcs})\big)^{\text{ar}(\sigma)}, \qquad \text{Feas}^{(m,n)}_\sigma : \ \big(\mathcal{P}^m(()S_{\rm hcs})\big)^{\text{ar}(\sigma)} \longrightarrow \mathcal{P}\big(\mathcal{P}^n(()S_{\rm hcs})\big),$$

and a closed convex cone $C_\sigma \subseteq V$ in the KPI monoid $(V, +, \leq)$ together with the valuation $\nu : S_{\rm hcs} \to V$ from Definition 2.14. A *Healthcare SuperHyperStructure of order* (m,n) is a tuple

$$\mathbf{HSHS} \;=\; \Big(S_{\rm hcs},\; \Sigma_{\rm hcs},\; \{\odot_\sigma^{(m,n)}\}_{\sigma\in\Sigma_{\rm hcs}},\; \nu,\; \mathrm{Obs},\; \{\mathrm{Feas}_\sigma^{(m,n)}\},\; \{C_\sigma\},\; \{D_\sigma^{(m)}\}\Big),$$

where each *superhyperoperation*

$$\odot_\sigma^{(m,n)} :\; \big(\mathcal{P}^{\,m}(()S_{\rm hcs})\big)^{\mathrm{ar}(\sigma)} \longrightarrow \mathcal{P}^{\,n}(()S_{\rm hcs})$$

satisfies, for all inputs $\vec{X} = (X_1, \ldots, X_{\mathrm{ar}(\sigma)})$:

(HSH1) **Feasible nonemptiness (lifted):** if $\vec{X} \in D_\sigma^{(m)}$ then $\odot_\sigma^{(m,n)}(\vec{X}) \neq \emptyset$.

(HSH2) **Policy compatibility (lifted):**

$$\odot_\sigma^{(m,n)}(\vec{X}) \;\subseteq\; \mathrm{Feas}_\sigma^{(m,n)}(\vec{X}).$$

(HSH3) **Atomic KPI admissibility:** for every $Y \in \odot_\sigma^{(m,n)}(\vec{X})$ and every $u \in \mathrm{At}(Y)$ there exist $s_i \in \mathrm{At}(X_i)$ with

$$\nu(u) \;\in\; \nu(s_1) + \cdots + \nu(s_{\mathrm{ar}(\sigma)}) \;+\; C_\sigma.$$

(HSH4) **Observational coherence (lifted):** there exists

$$H_\sigma^{(m,n)} :\; \big(\mathcal{P}^{\,m}(()\mathcal{O})\big)^{\mathrm{ar}(\sigma)} \longrightarrow \mathcal{P}^{\,n}(()\mathcal{O})$$

such that

$$\mathrm{Obs}^{\uparrow n}\big(\odot_\sigma^{(m,n)}(\vec{X})\big) \;\subseteq\; H_\sigma^{(m,n)}\big(\mathrm{Obs}^{\uparrow m}(X_1), \ldots, \mathrm{Obs}^{\uparrow m}(X_{\mathrm{ar}(\sigma)})\big).$$

When $(m,n) = (0,1)$ and $\odot_\sigma^{(0,1)} = \sigma^S$, this reduces to the Healthcare HyperStructure of Definition 2.14.

Example 2.23 (Regional ED surge activation from scenario ensembles: $(m,n) = (1,2)$)**.** Let $S_{\rm hcs}$ be ED network states $s = (q, b, r)$ with queue vector $q \in \mathbb{N}^L$, free beds $b \in \mathbb{N}$, and resources $r \in \mathbb{R}^d_{\geq 0}$. Let $V = \mathbb{R}^2_{\geq 0}$ and $\nu(s) = (\text{wait\_proxy}(s),\ \text{overtime\_proxy}(s))$. Consider a unary symbol $\sigma = \text{surge\_activate}$. Inputs are nonempty sets $X \in \mathcal{P}^{\,1}(()S_{\rm hcs}) = \mathcal{P}(S_{\rm hcs}) \setminus \{\emptyset\}$, e.g. *demand scenarios* collected over a short horizon.

Fix a finite family $\mathcal{T}$ of *surge tiers* (tiered policies: staffing call-ins, diversions, field triage). For $t \in \mathcal{T}$ let $\sigma_t^S : S_{\rm hcs} \to \mathcal{P}(S_{\rm hcs})$ be the tier-t hyperoperation (deterministic or set-valued under uncertainty), and let $\mathrm{Feas}_\sigma^{(1,2)}(X)$ encode systemwide regulatory/operational filters. Choose a KPI cone $C_\sigma = \{(x_1, x_2) \mid 0 \leq x_1 \leq W_{\max},\ 0 \leq x_2 \leq O_{\max}\}$.

Define the superhyperoperation

$$\odot_\sigma^{(1,2)}(X) \;:=\; \Big\{ \mathcal{Y} \subseteq S_{\rm hcs} \;\Big|\; \exists\, \mathcal{T}' \subseteq \mathcal{T} \text{ finite},\ \mathcal{Y} = \big\{\, u \in \textstyle\bigcup_{s\in X} \bigcup_{t\in\mathcal{T}'} \sigma_t^S(s) \,\big\} \;\wedge\; \mathcal{Y} \subseteq \mathrm{Feas}_\sigma^{(1,2)}(X) \Big\}.$$

Checks. If every $s \in X$ passes screening $D_\sigma^{(1)}$, some tier set $\mathcal{T}'$ yields a nonempty union, hence (HSH1). Inclusion by definition gives (HSH2). For any $Y \in \odot_\sigma^{(1,2)}(X)$ and $u \in \mathrm{At}(Y)$, there exist $s \in \mathrm{At}(X)$ and $t \in \mathcal{T}'$ with $u \in \sigma_t^S(s)$, so $\nu(u) - \nu(s) \in C_\sigma$ (HSH3). With Obs the live dashboard, let $H_\sigma^{(1,2)}$ be the level-2 collection of dashboards admissible after tier activation; then $\mathrm{Obs}^{\uparrow 2}(\odot_\sigma^{(1,2)}(X)) \subseteq H_\sigma^{(1,2)}(\mathrm{Obs}^{\uparrow 1}(X))$ (HSH4).

Example 2.24 (Multi-site OR block–swap consensus from proposal families: $(m,n) = (2,1)$)**.** Let $S_{\rm hcs}$ contain hospital-day states $s = (\mathcal{J}, B, \kappa)$ (cases $\mathcal{J}$, block map B, turnover/staff κ). Let $V = \mathbb{R}^2_{\geq 0}$ with $\nu(s) = (\text{overtime}(s),\ \text{idle\_time}(s))$. A single day is planned across sites; the input $X \in \mathcal{P}^{\,2}(()S_{\rm hcs})$ is a *family of feasible proposal sets* (one set per site/service).

Let Π be a reconciliation map that transforms a proposed timetable s into a *network-consistent* timetable $u = \Pi(s)$ (e.g. swapping blocks, synchronizing anesthesia teams), subject to policy $\mathrm{Feas}_\sigma^{(2,1)}(X)$ and KPI cone $C_\sigma = \{(x_1, x_2) \mid 0 \leq x_1 \leq O_{\max},\ 0 \leq x_2 \leq I_{\max}\}$. Define for unary $\sigma = \text{or\_consensus}$:

$$\odot_\sigma^{(2,1)}(X) \;:=\; \Big\{ u \in S_{\rm hcs} \;\Big|\; \exists\, Y \in X,\ \exists\, s \in Y :\ u = \Pi(s),\ u \in \mathrm{Feas}_\sigma^{(2,1)}(X),\ \nu(u) - \nu(s) \in C_\sigma \Big\}.$$

Checks. If some $Y \in X$ is feasible ($Y \in D_\sigma^{(2)}$), then $\odot_\sigma^{(2,1)}(X) \neq \emptyset$ (HSH1). Policy is enforced by membership in $\mathrm{Feas}_\sigma^{(2,1)}(X)$ (HSH2). Atomic admissibility holds since u is derived from an atom $s \in \mathrm{At}(Y)$ with bounded KPI drift (HSH3). If Obs returns case/room snapshots and KPIs, let $H_\sigma^{(2,1)}$ aggregate family-level observations into a single consensus snapshot; then (HSH4) holds.

Example 2.25 (Pharmacy procurement under lead-time uncertainty: $(m,n) = (1,1)$)**.** (cf. [40]) Let S_{hcs} be pharmacy states $s = (x, \hat{d}, L)$ (on-hand x, demand forecast $\hat{d}$, lead times L). Let $V = \mathbb{R}^2_{\geq 0}$ with $\nu(s) = (\mathrm{stockout\_risk}(s),\ \mathrm{holding\_cost}(s))$. An input $X \in \mathcal{P}^1(()S_{\mathrm{hcs}})$ collects *lead-time scenarios* consistent with current suppliers. Let $\mathcal{Q}(X)$ be the admissible order vectors under budget/service constraints evaluated *robustly* over X. Fix $C_\sigma = \{(r,h) \mid 0 \leq r \leq R_{\max},\ 0 \leq h \leq H_{\max}\}$.

For unary $\sigma = \mathrm{robust\_replenish}$ define

$$\odot_\sigma^{(1,1)}(X) \ := \ \Big\{ u = (x+q, \hat{d}, L') \in S_{\mathrm{hcs}} \ \Big| \ \exists\, s = (x, \hat{d}, L) \in X,\ \exists\, q \in \mathcal{Q}(X) : \\ u \in \mathrm{Feas}_\sigma^{(1,1)}(X) \ \wedge \ \nu(u) - \nu(s) \in C_\sigma \Big\}.$$

Checks. Nonemptiness holds when a robust order q exists (HSH1). Policy feasibility is built into $\mathrm{Feas}_\sigma^{(1,1)}$ (HSH2). Taking the witness $s \in \mathrm{At}(X)$ for u gives atomic KPI admissibility (HSH3). With Obs the inventory dashboard and $H_\sigma^{(1,1)}$ the set of post-order KPI projections across X, we obtain (HSH4).

Example 2.26 (Outpatient network template bank selection: $(m,n) = (0,2)$)**.** (cf. [41]) Let S_{hcs} contain clinic states $s = (\mathcal{A}, \rho, \theta)$ (appointments $\mathcal{A}$, no-show estimate ρ, resources θ). Let $V = \mathbb{R}^2_{\geq 0}$ with $\nu(s) = (\mathrm{expected\_wait}(s),\ \mathrm{overtime\_risk}(s))$. Fix a finite *template bank* $\mathfrak{T}$ (overbooking/capacity templates per hour/clinic) and let $\tau(s) \in \mathcal{P}(S_{\mathrm{hcs}})$ be the set of next states yielded by template $\tau \in \mathfrak{T}$. Let $C_\sigma = \{(w,o) \mid 0 \leq w \leq W_{\max},\ 0 \leq o \leq O_{\max}\}$.

For unary $\sigma = \mathrm{template\_select}$ and $s \in S_{\mathrm{hcs}}$, set

$$\odot_\sigma^{(0,2)}(s) \ := \ \Big\{ \mathcal{Y} \subseteq S_{\mathrm{hcs}} \ \Big| \ \exists\, \mathcal{T}' \subseteq \mathfrak{T} \text{ finite s.t. } \mathcal{Y} = \{\, u \in \tau(s) \mid \tau \in \mathcal{T}' \,\} \ \wedge \ \mathcal{Y} \subseteq \mathrm{Feas}_\sigma^{(0,2)}(s) \Big\}.$$

Checks. Screening $s \in D_\sigma^{(0)}$ ensures some admissible template, hence nonemptiness (HSH1). Inclusion into $\mathrm{Feas}_\sigma^{(0,2)}(s)$ gives (HSH2). For any $Y \in \odot_\sigma^{(0,2)}(s)$ and $u \in \mathrm{At}(Y)$, we have $u \in \tau(s)$ for some τ and $\nu(u) - \nu(s) \in C_\sigma$ (HSH3). With Obs the current schedule dashboard, let $H_\sigma^{(0,2)}$ collect the family of post-template dashboards to obtain (HSH4).

Theorem 2.27 (Healthcare SuperHyperStructure is a SuperHyperStructure)**.** *Let* **HSHS** *be as above. Then the* underlying family

$$\mathcal{SH}_{\mathrm{hcs}}^{(m,n)} \ := \ \Big(S_{\mathrm{hcs}},\ \Sigma_{\mathrm{hcs}},\ \{\odot_\sigma^{(m,n)}\}_{\sigma \in \Sigma_{\mathrm{hcs}}} \Big)$$

is an (m,n)-SuperHyperStructure on S_{hcs}.

Proof. Fix $\sigma \in \Sigma_{\mathrm{hcs}}$ and $k = \mathrm{ar}(\sigma)$. By Definition 2.22, the *type* of $\odot_\sigma^{(m,n)}$ is

$$\odot_\sigma^{(m,n)} : \ \big(\mathcal{P}^m(S_{\mathrm{hcs}})\big)^k \longrightarrow \mathcal{P}^n(S_{\mathrm{hcs}}).$$

Thus for every input $\vec{X} \in \big(\mathcal{P}^m(S_{\mathrm{hcs}})\big)^k$ the image $\odot_\sigma^{(m,n)}(\vec{X})$ is an element of $\mathcal{P}^n(S_{\mathrm{hcs}})$, i.e., a set at level n. Since this is true for every $\sigma \in \Sigma_{\mathrm{hcs}}$, the collection $\{\odot_\sigma^{(m,n)}\}_{\sigma \in \Sigma_{\mathrm{hcs}}}$ satisfies exactly the data required in the baseline definition of an (m,n)-SuperHyperStructure. Hence $\mathcal{SH}_{\mathrm{hcs}}^{(m,n)}$ is an (m,n)-SuperHyperStructure on S_{hcs}. □

3 Conclusion

In this paper, we explored Hyperstructures and Superhyperstructures within the domains of medicine and healthcare. These frameworks are expected to provide a more natural way to represent hierarchical structures in medicine and healthcare.

Looking ahead, we hope that future research will extend these concepts by incorporating frameworks such as Fuzzy Sets [42–44], Intuitionistic Fuzzy Sets [45, 46], Neutrosophic Sets [47, 48], Complex Neutrosophic Sets [49–51], Bipolar Neutrosophic Sets [52–54], Hesitant Fuzzy Sets [55, 55], HyperFuzzy Sets [56–58], and Plithogenic Sets [59–61].

We also anticipate that quantitative analyses will be carried out to further validate and enhance these theoretical developments. Furthermore, it is desirable that experimental studies and investigations using real datasets will also be pursued.

Funding

No external funding was received for this work.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this work.

Acknowledgments

We thank all colleagues, reviewers, and readers whose comments and questions have greatly improved this manuscript. We are also grateful to the authors of the works cited herein for providing the theoretical foundations that underpin our study. Finally, we appreciate the institutional and technical support that enabled this research.

Data Availability

This paper is theoretical and did not generate or analyze any empirical data. We welcome future studies that apply and test these concepts in practical settings.

Research Integrity

The author confirms that this manuscript is original, has not been published elsewhere, and is not under consideration by any other journal.

Use of Computational Tools

All proofs and derivations were performed manually; no computational software (e.g., Mathematica, SageMath, Coq) was used.

Code Availability

No code or software was developed for this study.

Ethical Approval

This research did not involve human participants or animals, and therefore did not require ethical approval.

Use of Generative AI and AI-Assisted Tools

We use generative AI and AI-assisted tools for tasks such as English grammar checking, and We do not employ them in any way that violates ethical standards.

Supplementary Information

No supplementary materials accompany this paper.

Disclaimer

The ideas presented here are theoretical and have not yet been validated through empirical testing. While we have strived for accuracy and proper citation, inadvertent errors may remain. Readers should verify any referenced material independently. The opinions expressed are those of the authors and do not necessarily reflect the views of their institutions.

References

[1] M Al Tahan and Bijan Davvaz. Weak chemical hyperstructures associated to electrochemical cells. *Iranian Journal of Mathematical Chemistry*, 9(1):65–75, 2018.

[2] Bijan Davvaz and Irina Cristea. Fuzzy algebraic hyperstructures. *Studies in Fuzziness and soft computing*, 321:38–46, 2015.

[3] Sultan Yamak, Osman Kazancı, and Bijan Davvaz. Soft hyperstructure. *Computers & Mathematics with Applications*, 62(2):797–803, 2011.

[4] Bijan Davvaz and Thomas Vougiouklis. *Walk Through Weak Hyperstructures, A: Hv-structures*. World Scientific, 2018.

[5] Sang-Cho Chung. Chemical hyperstructures for ozone depletion. *Journal of the Chungcheong Mathematical Society*, 32(4):491–508, 2019.

[6] Sang-Cho Chung and Kang Moon Chun. Chemical hyperstructures for stratospheric ozone depletion. *Journal of the Chungcheong Mathematical Society*, 33(4):469–487, 2020.

[7] Kang Moon Chun. Chemical hyperstructures of chemical reactions for iron and indium. *Journal of the Chungcheong Mathematical Society*, 27(2):319–319, 2014.

[8] Florentin Smarandache. Superhyperstructure & neutrosophic superhyperstructure, 2024. Accessed: 2024-12-01.

[9] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[10] Prabakaran Raghavendran and Tharmalingam Gunasekar. Optimizing organ transplantation success using neutrosophic superhyperstructure and artificial intelligence. *Volume IV*, page 117.

[11] Florentin Smarandache. *SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions*. Infinite Study, 2023.

[12] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[13] F. Smarandache. Introduction to superhyperalgebra and neutrosophic superhyperalgebra. *Journal of Algebraic Hyperstructures and Logical Algebras*, 2022.

[14] Takaaki Fujita and Florentin Smarandache. A unified framework for u-structures and functorial structure: Managing super, hyper, superhyper, tree, and forest uncertain over/under/off models. *Neutrosophic Sets and Systems*, 91:337–380, 2025.

[15] Souzana Vougioukli. Helix-hyperoperations on lie-santilli admissibility. *Algebras Groups and Geometries*, 2023.

[16] Souzana Vougioukli. Helix hyperoperation in teaching research. *Science & Philosophy*, 8(2):157–163, 2020.

[17] Thomas Vougiouklis. *Hyperstructures and their representations*. Hadronic Press, 1994.

[18] Piergiulio Corsini and Violeta Leoreanu. *Applications of hyperstructure theory*, volume 5. Springer Science & Business Media, 2013.

[19] Takaaki Fujita. Chemical hyperstructures, superhyperstructures, and shv-structures: Toward a generalized framework for hierarchical chemical modeling. *ChemRxiv*, 2025.

[20] Sara A Buckman, Isaiah R Turnbull, and John E Mazuski. Empiric antibiotics for sepsis. *Surgical infections*, 19(2):147–154, 2018.

[21] Kamlesh Khunti, Francesco Giorgino, Lori Berard, Didac Mauricio, and Stewart B Harris. The importance of the initial period of basal insulin titration in people with diabetes. *Diabetes, Obesity and Metabolism*, 22(5):722–733, 2020.

[22] John Walsh, Ruth Roberts, Timothy S Bailey, and Lutz Heinemann. Insulin titration guidelines for patients with type 1 diabetes: it is about time! *Journal of Diabetes Science and Technology*, 17(4):1066–1076, 2023.

[23] Shashvat M Desai, Randy Bravo, Joshua Catapano, Angelina Cooper, and Ashutosh P Jadhav. Abstract number-205: Impact of automated neuroimaging triage platform on time metrics at a thrombectomy capable stroke center. *Stroke: Vascular and Interventional Neurology*, 3:e12630, 2023.

[24] Anna Gundlund, Laila Staerk, Emil Loldrup Fosbøl, K Gadsbøll, Caroline Sindet-Pedersen, Anders Nissen Bonde, Gunnar H Gislason, and JB Olesen. Initiation of anticoagulation in atrial fibrillation: which factors are associated with choice of anticoagulant? *Journal of Internal Medicine*, 282(2):164–174, 2017.

[25] Nihar R Desai, Alexis A Krumme, Sebastian Schneeweiss, William H Shrank, Gregory Brill, Edmund J Pezalla, Claire M Spettell, Troyen A Brennan, Olga S Matlin, Jerry Avorn, et al. Patterns of initiation of oral anticoagulants in patients with atrial fibrillation-quality and cost implications. *The American journal of medicine*, 127(11):1075–1082, 2014.

[26] Abbas Mardani, Robert E Hooker, Seckin Ozkul, Sun Yifan, Mehrbakhsh Nilashi, Hamed Zamani Sabzi, and Goh Chin Fei. Application of decision making and fuzzy sets theory to evaluate the healthcare and medical problems: a review of three decades of research with recent developments. *Expert Systems with Applications*, 137:202–231, 2019.

[27] Güney Gürsel. Healthcare, uncertainty, and fuzzy logic. *Digital Medicine*, 2(3):101–112, 2016.

[28] Thanh Nguyen, Abbas Khosravi, Douglas Creighton, and Saeid Nahavandi. Classification of healthcare data using genetic fuzzy logic system and wavelets. *Expert Systems with Applications*, 42(4):2184–2197, 2015.

[29] Florentin Smarandache and Daniela Gifu. Soft sets extensions: Innovating healthcare claims analysis. *Available at SSRN 5015972*, 2024.

[30] Joseph L Nates, Mark Nunnally, Ruth Kleinpell, Sandralee Blosser, Jonathan Goldner, Barbara Birriel, Clara S Fowler, Diane Byrum, William Scherer Miles, Heatherlee Bailey, et al. Icu admission, discharge, and triage guidelines: a framework to enhance clinical operations, development of institutional policies, and further research. *Critical care medicine*, 44(8):1553–1602, 2016.

[31] Wanyi Chen, Benjamin Linthicum, Nilay Tanik Argon, Thomas Bohrmann, Kenneth Lopiano, Abhi Mehrotra, Debbie Travers, and Serhan Ziya. The effects of emergency department crowding on triage and hospital admission decisions. *The American journal of emergency medicine*, 38(4):774–779, 2020.

[32] Wilton C Levine and Peter F Dunn. Optimizing operating room scheduling. *Anesthesiology clinics*, 33(4):697–711, 2015.

[33] Aida Jebali, Atidel B Hadj Alouane, and Pierre Ladet. Operating rooms scheduling. *International Journal of Production Economics*, 99(1-2):52–62, 2006.

[34] Lu He, Sreenath Chalil Madathil, Amrita Oberoi, Greg Servis, and Mohammad T Khasawneh. A systematic review of research design and modeling techniques in inpatient bed management. *Computers & Industrial Engineering*, 127:451–466, 2019.

[35] EK Lee, Z Wang, and A Shapoval. Strategies for inpatient bed management. *J. Health Med. Inf*, 9(02), 2018.

[36] Christos Bialas, Andreas Revanoglou, and Vicky Manthou. Improving hospital pharmacy inventory management using data segmentation. *American Journal of Health-System Pharmacy*, 77(5):371–377, 2020.

[37] Rachel O'Connor, Sang Won Yoon, and Soongeol Kwon. Analysis and optimization of replenishment process for robotic dispensing system in a central fill pharmacy. *Computers & Industrial Engineering*, 154:107116, 2021.

[38] Sangbok Lee, Daiki Min, Jong-hyun Ryu, and Yuehwern Yih. A simulation study of appointment scheduling in outpatient clinics: Open access and overbooking. *Simulation*, 89(12):1459–1473, 2013.

[39] Linda R LaGanga and Stephen R Lawrence. Clinic overbooking to improve patient access and increase provider productivity. *Decision Sciences*, 38(2):251–276, 2007.

[40] Marius Călin Chereches, Cristian Olimpiu Popa, and Hajnal Finta. The dynamics of food for special medical purposes (fsmps) utilization in cancer care: from doctor recommendations to online pharmacy procurement. *Frontiers in Pharmacology*, 15:1393784, 2024.

[41] Ahmed Baita Garko and Usman Mahmud. Design and implementation of outpatient management system. *International Journal of Advanced Academic Research*, 3(6):13–22, 2017.

[42] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[43] Talal Al-Hawary. Complete fuzzy graphs. *International Journal of Mathematical Combinatorics*, 4:26, 2011.

[44] John N Mordeson and Premchand S Nair. *Fuzzy graphs and fuzzy hypergraphs*, volume 46. Physica, 2012.

[45] Krassimir T Atanassov. Circular intuitionistic fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 39(5):5981–5986, 2020.

[46] Krassimir T Atanassov and G Gargov. *Intuitionistic fuzzy logics*. Springer, 2017.

[47] Haibin Wang, Florentin Smarandache, Yanqing Zhang, and Rajshekhar Sunderraman. *Single valued neutrosophic sets*. Infinite study, 2010.

[48] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Single valued neutrosophic graphs. *Journal of New theory*, (10):86–101, 2016.

[49] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Complex neutrosophic graphs of type. *Collected Papers. Volume VI: On Neutrosophic Theory and Applications*, page 204, 2022.

[50] Naveed Yaqoob and Muhammad Akram. *Complex neutrosophic graphs*. Infinite Study, 2018.

[51] Anam Luqman, Muhammad Akram, and Florentin Smarandache. Complex neutrosophic hypergraphs: New social network models. *Algorithms*, 12:234, 2019.

[52] Irfan Deli, Mumtaz Ali, and Florentin Smarandache. Bipolar neutrosophic sets and their application based on multi-criteria decision making problems. *2015 International Conference on Advanced Mechatronic Systems (ICAMechS)*, pages 249–254, 2015.

[53] Vakkas Ulucay, Irfan Deli, and Mehmet Sahin. Similarity measures of bipolar neutrosophic sets and their application to multiple criteria decision making. *Neural Computing and Applications*, 29:739–748, 2018.

[54] Mumtaz Ali, Le Hoang Son, Irfan Deli, and Nguyen Dang Tien. Bipolar neutrosophic soft sets and applications in decision making. *Journal of Intelligent & Fuzzy Systems*, 33(6):4077–4087, 2017.

[55] Vicenç Torra and Yasuo Narukawa. On hesitant fuzzy sets and decision. In *2009 IEEE international conference on fuzzy systems*, pages 1378–1382. IEEE, 2009.

[56] Yong Lin Liu, Hee Sik Kim, and J. Neggers. Hyperfuzzy subsets and subgroupoids. *J. Intell. Fuzzy Syst.*, 33:1553–1562, 2017.

[57] Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.

[58] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[59] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Plithogenic Graphs*. Infinite Study, 2020.

[60] Nivetha Martin. Plithogenic swara-topsis decision making on food processing methods with different normalization techniques. *Advances in Decision Making*, 69, 2022.

[61] Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.