A note on the necessary and sufficient conditions to matrix summability of infinite series

ABSTRACT. Earlier works have established the conditions under which the series $\sum a_n\gamma_n$ is $\phi - \mid F,\mu;\tau\mid_k$ summable, provided that the series $\sum a_n$ is summable by $\phi - \mid E,\mu;\tau\mid$. In this study, we present the necessary and sufficient criteria ensuring $\phi-\mid F,\mu;\tau\mid$ summability of $\sum a_n\gamma_n$ whenever $\sum a_n$ is summable by $\phi-\mid E,\mu;\tau\mid_k$, where $E=(e_{nv})$ and $F=(f_{nv})$ denote two positive normal matrices, $k\geq 1$, $\tau\geq 0$, and the inequality $-\mu(\tau k+k-1)+k>0$ holds.
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MSC (2020): 40D25; 40F05; 40G99

1 Introduction

Let $\sum a_n$ be an infinite series (ISs), and denote its sequence of partial sums by (s_n). Let $E=(e_{nv})$ be a lower triangular matrix of nonzero diagonal entries (normal matrix). The matrix E defines a transformation from a sequence $s=(s_n)$ to the sequence $Es=(E_n(s))$, given by

$$E_n(s)=\sum_{v=0}^{n} e_{nv}s_v, \quad n=0,1,...$$

Suppose $E=(e_{nv})$ is a normal matrix. The lower semimatrices $\bar{E}=(\bar{e}_{nv})$ and $\hat{E}=(\hat{e}_{nv})$ are then introduced as follows:

$$\bar{e}_{nv}=\sum_{i=v}^{n} e_{ni}, \quad n,v=0,1,...$$

$$\hat{e}_{00} = \bar{e}_{00} = e_{00}, \quad \hat{e}_{nv} = \bar{e}_{nv} - \bar{e}_{n-1,v}, \quad n = 1, 2, ...$$

and

$$\bar{\Delta} E_n(s) = E_n(s) - E_{n-1}(s) = \sum_{v=0}^{n} \hat{e}_{nv} a_v \tag{1}$$

Let E be a normal matrix. Its inverse is represented by $E' = (e'_{nv})$. Similarly, $\hat{E} = (\hat{e}_{nv})$ is a normal matrix that admits a two-sided inverse $\hat{E}' = (\hat{e}'_{nv})$, which is again normal (see [1]).

Consider a sequence (ϕ_n) of positive terms. The series $\sum a_n$ is said to be $\phi - \mid E, \mu; \tau \mid_k$ summable whenever $k \geq 1$, $\tau \geq 0$, and $\mu \in \mathbb{R}$, provided that [2]

$$\sum_{n=1}^{\infty} \phi_n^{\mu(\tau k + k - 1)} \mid E_n(s) - E_{n-1}(s) \mid^k < \infty.$$

For $\phi_n = n$ for all values of n, $\mu = 1$ and $\tau = 0$, we get $|E|_k$ summability method [3].

The space l_k is defined as $l_k := \left\{ x = (x_j) : \sum |x_j|^k < \infty \right\}$.

Lemma [4] Let $k \geq 1$, $p \geq 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, $\frac{1}{p} + \frac{1}{p'} = 1$. Let

$$j_n = \sum_{m=0}^{\infty} h_{nm} q_m, \quad n \geq 0$$

$$v_m = \sum_{n=0}^{\infty} h_{nm} u_n, \quad m \geq 0$$

It follows that $j \in l_p$ whenever $q \in l_k$, iff $v \in l_{k'}$ whenever $u \in l_{p'}$.

2 Main Result

Recently, studies on the application of absolute summability and absolute matrix summability to different factored ISs and some well-known classes of sequence have been presented. Before giving the main result, let's mention about the some of these studies. Bor [5,6], Bor and Agarwal [7,8], Bor and Mohapatra [9] used the different class of sequences to get theorems on absolute summability. Karakaş [10], Kartal [11], Özarslan and Şakar [12] established sufficient conditions for the absolute Riesz summability of the ISs. Karakaş [13], Özarslan [14–16], Özarslan and Kartal [17], Özarslan, Şakar and Kartal [18] got general theorems on absolute matrix summability methods. Kartal [19, 20], Özarslan [21, 22] used almost increasing, δ-quasi-monotone and quasi power increasing sequences to obtain sufficient conditions for absolute summability of the ISs. Recently, Kartal [23] established the conditions for $\phi-\mid F,\mu;\tau\mid_k$ summability of the series $\sum a_n\gamma_n$ under the assumption that $\sum a_n$ is $\phi-\mid E,\mu;\tau\mid$ summable, which serves as the main motivation for this work. The purpose of the present study is to derive the necessary and sufficient conditions for $\phi-\mid F,\mu;\tau\mid$ summability of $\sum a_n\gamma_n$ whenever $\sum a_n$ is $\phi-\mid E,\mu;\tau\mid_k$ summable, as detailed below.

Theorem Let $k\geq 1$, $\tau\geq 0$ and $-\mu(\tau k+k-1)+k>0$. Consider two positive normal matrices $E=(e_{nv})$ and $F=(f_{nv})$ which satisfy

$$e_{nn}-e_{n+1,n}=O(e_{nn}e_{n+1,n+1}) \tag{2}$$

$$\overline{f}_{n0}=1,\quad n=0,1,...$$

$$\sum_{v=r+2}^{n}|\hat{f}_{nv}\hat{e}'_{vr}\gamma_v|=O\left(\left|\hat{f}_{n,r+1}\gamma_{r+1}\right|\right). \tag{3}$$

The series $\sum a_n\gamma_n$ is $\phi-\mid F,\mu;\tau\mid$ summable whenever $\sum a_n$ is $\phi-\mid E,\mu;\tau\mid_k$ summable, if and only if

$$\left\{\phi_v^{\frac{-\mu(\tau k+k-1)}{k}}\left(\frac{f_{vv}}{e_{vv}}|\gamma_v|+\sum_{n=v+1}^{\infty}\left(\frac{\left|\Delta_v\left(\hat{f}_{nv}\gamma_v\right)\right|}{e_{vv}}+\left|\hat{f}_{n,v+1}\gamma_{v+1}\right|\right)+\sum_{n=v+2}^{\infty}\left|\hat{f}_{n,v+1}\gamma_{v+1}\right|\right)\right\}\in l_{k'}$$

3 Proof of the theorem

Consider the sequences (I_n) and (U_n) representing the E-transform of $\sum a_n$ and the F-transform of $\sum a_n\gamma_n$. Then, according to (1), we have

$$x_n=\bar{\Delta}I_n=\sum_{v=0}^{n}\hat{e}_{nv}a_v\quad and\quad y_n=\bar{\Delta}U_n=\sum_{v=0}^{n}\hat{f}_{nv}a_v\gamma_v. \tag{4}$$

On the basis of the equalities in (4), we obtain $a_v=\sum_{r=0}^{v}\hat{e}'_{vr}x_r$ and $y_n=\sum_{v=0}^{n}\hat{f}_{nv}\gamma_v\sum_{r=0}^{v}\hat{e}'_{vr}x_r$.

Since $\hat{f}_{n0}=\bar{f}_{n0}-\bar{f}_{n-1,0}=0$, we have

$$\begin{aligned}y_n &= \sum_{v=1}^{n}\hat{f}_{nv}\gamma_v\sum_{r=0}^{v}\hat{e}'_{vr}x_r\\ &= \sum_{v=1}^{n}\hat{f}_{nv}\gamma_v\hat{e}'_{vv}x_v+\sum_{v=1}^{n}\hat{f}_{nv}\gamma_v\hat{e}'_{v,v-1}x_{v-1}+\sum_{v=1}^{n}\hat{f}_{nv}\gamma_v\sum_{r=0}^{v-2}\hat{e}'_{vr}x_r\\ &= \hat{f}_{nn}\gamma_n\hat{e}'_{nn}x_n+\sum_{v=1}^{n-1}\left(\hat{f}_{nv}\gamma_v\hat{e}'_{vv}+\hat{f}_{n,v+1}\gamma_{v+1}\hat{e}'_{v+1,v}\right)x_v\\ &+ \sum_{r=0}^{n-2}x_r\sum_{v=r+2}^{n}\hat{f}_{nv}\gamma_v\hat{e}'_{vr}.\end{aligned} \tag{5}$$

For δ_{nv}(Kronecker delta), with the help of the equality $\sum_{k=v}^{n} \hat{e}'_{nk}\hat{e}_{kv} = \delta_{nv}$, we get

$$\begin{aligned}
\hat{f}_{nv}\gamma_v \hat{e}'_{vv} + \hat{f}_{n,v+1}\gamma_{v+1}\hat{e}'_{v+1,v} &= \frac{\hat{f}_{nv}\gamma_v}{\hat{e}_{vv}} + \hat{f}_{n,v+1}\gamma_{v+1}\left(-\frac{\hat{e}_{v+1,v}}{\hat{e}_{vv}\hat{e}_{v+1,v+1}}\right) \\
&= \frac{\hat{f}_{nv}\gamma_v}{e_{vv}} - \hat{f}_{n,v+1}\gamma_{v+1}\frac{(\bar{e}_{v+1,v} - \bar{e}_{vv})}{e_{vv}e_{v+1,v+1}} \\
&= \frac{\hat{f}_{nv}\gamma_v}{e_{vv}} - \hat{f}_{n,v+1}\gamma_{v+1}\frac{(e_{v+1,v+1} + e_{v+1,v} - e_{vv})}{e_{vv}e_{v+1,v+1}} \\
&= \frac{\Delta_v(\hat{f}_{nv}\gamma_v)}{e_{vv}} + \hat{f}_{n,v+1}\gamma_{v+1}\frac{(e_{vv} - e_{v+1,v})}{e_{vv}e_{v+1,v+1}}.
\end{aligned}$$

Substituting this equality into (5), we obtain

$$\begin{aligned}
y_n &= \frac{f_{nn}\gamma_n}{e_{nn}}x_n + \sum_{v=1}^{n-1}\frac{\Delta_v(\hat{f}_{nv}\gamma_v)}{e_{vv}}x_v + \sum_{v=1}^{n-1}\hat{f}_{n,v+1}\gamma_{v+1}\frac{(e_{vv} - e_{v+1,v})}{e_{vv}e_{v+1,v+1}}x_v \\
&+ \sum_{r=0}^{n-2}x_r\sum_{v=r+2}^{n}\hat{f}_{nv}\gamma_v\hat{e}'_{vr}.
\end{aligned}$$

Let $X_v = \phi_v^{\frac{\mu(\tau k+k-1)}{k}} x_v$. Consider the sequence (h_{nv}) defined by:

$$h_{nv} = \begin{cases}
\phi_v^{\frac{-\mu(\tau k+k-1)}{k}}\left(\frac{\Delta_v(\hat{f}_{nv}\gamma_v)}{e_{vv}} + \hat{f}_{n,v+1}\gamma_{v+1}\frac{(e_{vv}-e_{v+1,v})}{e_{vv}e_{v+1,v+1}} + \sum_{r=v+2}^{n}\hat{f}_{nr}\gamma_r\hat{e}'_{rv}\right), & 1 \le v \le n-2 \\
\phi_v^{\frac{-\mu(\tau k+k-1)}{k}}\left(\frac{\Delta_v(\hat{f}_{nv}\gamma_v)}{e_{vv}} + \hat{f}_{n,v+1}\gamma_{v+1}\frac{(e_{vv}-e_{v+1,v})}{e_{vv}e_{v+1,v+1}}\right) & v = n-1 \\
\phi_v^{\frac{-\mu(\tau k+k-1)}{k}}\frac{\hat{f}_{nv}\gamma_v}{e_{vv}} & v = n \\
0, & v > n
\end{cases}$$

It follows that

$$y_n = \sum_{v=1}^{\infty} h_{nv}X_v.$$

A necessary and sufficient condition for the series $\sum a_n\gamma_n$ to be $\phi - \mid F, \mu; \tau \mid$ summable, assuming that $\sum a_n$ is $\phi - \mid E, \mu; \tau \mid_k$ summable, is

$$\sum |y_n| < \infty \quad \text{whenever} \quad \sum |X_n|^k < \infty. \tag{6}$$

According to the Lemma, (6) is valid iff $\sum_{v=1}^{\infty}\left|\sum_{n=v}^{\infty}h_{nv}u_n\right|^{k'} < \infty$ whenever $u_n = O(1)$. It then follows from (2) and (3) that this inequality holds if and only if

$$\left\{\phi_v^{\frac{-\mu(\tau k+k-1)}{k}}\left(\frac{f_{vv}}{e_{vv}}|\gamma_v| + \sum_{n=v+1}^{\infty}\left(\frac{\left|\Delta_v\left(\hat{f}_{nv}\gamma_v\right)\right|}{e_{vv}} + \left|\hat{f}_{n,v+1}\gamma_{v+1}\right|\right) + \sum_{n=v+2}^{\infty}\left|\hat{f}_{n,v+1}\gamma_{v+1}\right|\right)\right\} \in l_{k'}$$

is satisfied. This completes the proof of Theorem.

4 Conclusion

This paper derives the necessary and sufficient criteria for the $\phi - \mid F, \mu; \tau \mid$ summability of the series $\sum a_n \gamma_n$ in the case where $\sum a_n$ is summable $\phi - \mid E, \mu; \tau \mid_k$. In particular, if $\phi_n = n$ for all n, $\mu = 1$, and $\tau = 0$, the corresponding conditions describe $\mid F \mid$ summability of $\sum a_n \gamma_n$ whenever $\sum a_n$ is summable $\mid E \mid_k$. Moreover, the results presented here may encourage further exploration of related problems under other summability methods.

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