A recent note on the necessary and sufficient conditions to matrix summability of infinite series

Abstract In [1], the conditions for $\varphi - \mid D,\beta;\delta \mid_k$ summability of the series $\sum a_n\gamma_n$ whenever the series $\sum a_n$ is summable $\varphi - \mid C,\beta;\delta \mid$ have already been obtained. In the present paper the necessary and sufficient conditions for $\varphi - \mid D,\beta;\delta \mid$ summability of the series $\sum a_n\gamma_n$ whenever the series $\sum a_n$ is summable $\varphi - \mid C,\beta;\delta \mid_k$, where $C = (c_{nv})$ and $D = (d_{nv})$ are two positive normal matrices, $k \geq 1$, $\delta \geq 0$ and $-\beta(\delta k + k - 1) + k > 0$ are given.

Keywords absolute matrix summability · infinite series · matrix transformations · sequence spaces · summability factors

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1 Introduction

Let $\sum a_n$ be an infinite series with its partial sums (s_n). Let $C = (c_{nv})$ be a normal matrix which means a lower triangular matrix of nonzero diagonal entries. Then C defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $Cs = (C_n(s))$, where

$$C_n(s) = \sum_{v=0}^{n} c_{nv}s_v, \quad n = 0, 1, ...$$

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Let $C = (c_{nv})$ be a normal matrix, then two lower semimatrices $\bar{C} = (\bar{c}_{nv})$ and $\hat{C} = (\hat{c}_{nv})$ are defined as follows:

$$\bar{c}_{nv} = \sum_{i=v}^{n} c_{ni}, \quad n, v = 0, 1, ...$$

$$\hat{c}_{00} = \bar{c}_{00} = c_{00}, \quad \hat{c}_{nv} = \bar{c}_{nv} - \bar{c}_{n-1,v}, \quad n = 1, 2, ...$$

and

$$\bar{\Delta} C_n(s) = C_n(s) - C_{n-1}(s) = \sum_{v=0}^{n} \hat{c}_{nv} a_v \tag{1}$$

If C is a normal matrix, then $C' = (c'_{nv})$ denotes the inverse of C, and $\hat{C} = (\hat{c}_{nv})$ is a normal matrix and it has two-sided inverse $\hat{C}' = (\hat{c}'_{nv})$ which is also normal (see [2]).

Let (φ_n) be a sequence of positive numbers. The series $\sum a_n$ is said to be summable $\varphi - \mid C, \beta; \delta \mid_k$, $k \geq 1$, $\delta \geq 0$ and β is a real number, if [3]

$$\sum_{n=1}^{\infty} \varphi_n^{\beta(\delta k + k - 1)} \mid C_n(s) - C_{n-1}(s) \mid^k < \infty$$

For $\varphi_n = n$ for all values of n, $\beta = 1$ and $\delta = 0$, we get $|C|_k$ summability method [4].

Let l_k denotes the set of sequence such that $l_k := \left\{ x = (x_j) : \sum |x_j|^k < \infty \right\}$.

Lemma 1 [5] *Let $k \geq 1$, $p \geq 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, $\frac{1}{p} + \frac{1}{p'} = 1$. Let*

$$d_n = \sum_{m=0}^{\infty} h_{nm} q_m, \quad n \geq 0$$

$$v_m = \sum_{n=0}^{\infty} h_{nm} u_n, \quad m \geq 0$$

In order that $d \in l_p$ whenever $q \in l_k$ if and only if $v \in l_{k'}$ whenever $u \in l_{p'}$.

2 Main Result

Recently, studies on the application of absolute summability and absolute matrix summability to different factored infinite series and some well-known classes of sequence have been presented. Before giving the main result, let's mention about the some of these studies. Bor [6, 7], Bor and Agarwal [8, 9], Bor and Mohapatra [10] used the different class of sequences to get theorems on absolute summability. Karakaş [11], Kartal [12], Özarslan and Şakar [13] obtained the sufficient conditions for absolute Riesz summability of the infinite series. Karakaş [14], Özarslan [15–17], Özarslan and Kartal [18], Özarslan, Şakar and Kartal [19] got general theorems on absolute matrix summability methods. Kartal [20–22], Özarslan [23–25] used almost increasing, δ-quasi-monotone and quasi power increasing sequences to obtain sufficient conditions for absolute summability of the infinite series. Shortly before, in [1], Kartal obtained the conditions for $\varphi - \mid D, \beta; \delta \mid_k$ summability of the series $\sum a_n \gamma_n$ whenever the series $\sum a_n$ is summable $\varphi - \mid C, \beta; \delta \mid$. The aim of the present article is to obtain the necessary and sufficient conditions for $\varphi - \mid D, \beta; \delta \mid$ summability of the series $\sum a_n \gamma_n$ whenever the series $\sum a_n$ is summable $\varphi - \mid C, \beta; \delta \mid_k$ as in the following.

Theorem 1 *Let $k \geq 1$, $\delta \geq 0$ and $-\beta(\delta k + k - 1) + k > 0$. Let $C = (c_{nv})$ and $D = (d_{nv})$ be two positive normal matrices satisfy*

$$c_{nn} - c_{n+1,n} = O(c_{nn} c_{n+1,n+1}) \tag{2}$$

$$\bar{d}_{n0} = 1, \quad n = 0, 1, ...$$

$$\sum_{v=r+2}^{n} |\hat{d}_{nv} \hat{c}'_{vr} \gamma_v| = O\left(\left|\hat{d}_{n,r+1} \gamma_{r+1}\right|\right) \tag{3}$$

Then $\sum a_n \gamma_n$ is summable $\varphi - \mid D, \beta; \delta \mid$ whenever $\sum a_n$ is summable $\varphi - \mid C, \beta; \delta \mid_k$ if and only if

$$\left\{ \varphi_v^{\frac{-\beta(\delta k + k - 1)}{k}} \left(\frac{d_{vv}}{c_{vv}} |\gamma_v| + \sum_{n=v+1}^{\infty} \left(\frac{\left|\Delta_v \left(\hat{d}_{nv} \gamma_v\right)\right|}{c_{vv}} + \left|\hat{d}_{n,v+1} \gamma_{v+1}\right| \right) + \sum_{n=v+2}^{\infty} \left|\hat{d}_{n,v+1} \gamma_{v+1}\right| \right) \right\} \in l_{k'}$$

Proof of Theorem 1

Let (I_n) and (U_n) denote C-transform and D-transform of the series $\sum a_n$ and $\sum a_n \gamma_n$. By (1), we get

$$x_n = \bar{\Delta} I_n = \sum_{v=0}^{n} \hat{c}_{nv} a_v \quad \textit{and} \quad y_n = \bar{\Delta} U_n = \sum_{v=0}^{n} \hat{d}_{nv} a_v \gamma_v \tag{4}$$

By the equalities in (4), we obtain $a_v = \sum_{r=0}^{v} \hat{c}'_{vr} x_r$ and $y_n = \sum_{v=0}^{n} \hat{d}_{nv} \gamma_v \sum_{r=0}^{v} \hat{c}'_{vr} x_r$.

Since $\hat{d}_{n0} = \bar{d}_{n0} - \bar{d}_{n-1,0} = 0$, we have

$$\begin{aligned} y_n &= \sum_{v=1}^{n} \hat{d}_{nv} \gamma_v \sum_{r=0}^{v} \hat{c}'_{vr} x_r \\ &= \sum_{v=1}^{n} \hat{d}_{nv} \gamma_v \hat{c}'_{vv} x_v + \sum_{v=1}^{n} \hat{d}_{nv} \gamma_v \hat{c}'_{v,v-1} x_{v-1} + \sum_{v=1}^{n} \hat{d}_{nv} \gamma_v \sum_{r=0}^{v-2} \hat{c}'_{vr} x_r \end{aligned}$$

$$= \hat{d}_{nn}\gamma_n \hat{c}'_{nn} x_n + \sum_{v=1}^{n-1} \left(\hat{d}_{nv}\gamma_v \hat{c}'_{vv} + \hat{d}_{n,v+1}\gamma_{v+1}\hat{c}'_{v+1,v} \right) x_v$$

$$+ \sum_{r=0}^{n-2} x_r \sum_{v=r+2}^{n} \hat{d}_{nv}\gamma_v \hat{c}'_{vr} \qquad (5)$$

For δ_{nv}(Kronecker delta), by using the equality $\sum_{k=v}^{n} \hat{c}'_{nk}\hat{c}_{kv} = \delta_{nv}$, we get

$$\begin{aligned}
\hat{d}_{nv}\gamma_v \hat{c}'_{vv} + \hat{d}_{n,v+1}\gamma_{v+1}\hat{c}'_{v+1,v} &= \frac{\hat{d}_{nv}\gamma_v}{\hat{c}_{vv}} + \hat{d}_{n,v+1}\gamma_{v+1}\left(-\frac{\hat{c}_{v+1,v}}{\hat{c}_{vv}\hat{c}_{v+1,v+1}} \right) \\
&= \frac{\hat{d}_{nv}\gamma_v}{c_{vv}} - \hat{d}_{n,v+1}\gamma_{v+1}\frac{(\bar{c}_{v+1,v} - \bar{c}_{vv})}{c_{vv}c_{v+1,v+1}} \\
&= \frac{\hat{d}_{nv}\gamma_v}{c_{vv}} - \hat{d}_{n,v+1}\gamma_{v+1}\frac{(c_{v+1,v+1} + c_{v+1,v} - c_{vv})}{c_{vv}c_{v+1,v+1}} \\
&= \frac{\Delta_v(\hat{d}_{nv}\gamma_v)}{c_{vv}} + \hat{d}_{n,v+1}\gamma_{v+1}\frac{(c_{vv} - c_{v+1,v})}{c_{vv}c_{v+1,v+1}}
\end{aligned}$$

When we write the above equality in (5), we get

$$y_n = \frac{d_{nn}\gamma_n}{c_{nn}} x_n + \sum_{v=1}^{n-1} \frac{\Delta_v(\hat{d}_{nv}\gamma_v)}{c_{vv}} x_v + \sum_{v=1}^{n-1} \hat{d}_{n,v+1}\gamma_{v+1}\frac{(c_{vv} - c_{v+1,v})}{c_{vv}c_{v+1,v+1}} x_v$$

$$+ \sum_{r=0}^{n-2} x_r \sum_{v=r+2}^{n} \hat{d}_{nv}\gamma_v \hat{c}'_{vr}$$

Let $X_v = \varphi_v^{\frac{\beta(\delta k+k-1)}{k}} x_v$. If we define the sequence (h_{nv}) as in the following:

$$h_{nv} = \begin{cases}
\varphi_v^{\frac{-\beta(\delta k+k-1)}{k}} \left(\frac{\Delta_v(\hat{d}_{nv}\gamma_v)}{c_{vv}} + \hat{d}_{n,v+1}\gamma_{v+1}\frac{(c_{vv}-c_{v+1,v})}{c_{vv}c_{v+1,v+1}} + \sum_{r=v+2}^{n} \hat{d}_{nr}\gamma_r \hat{c}'_{rv} \right), & 1 \le v \le n-2 \\
\varphi_v^{\frac{-\beta(\delta k+k-1)}{k}} \left(\frac{\Delta_v(\hat{d}_{nv}\gamma_v)}{c_{vv}} + \hat{d}_{n,v+1}\gamma_{v+1}\frac{(c_{vv}-c_{v+1,v})}{c_{vv}c_{v+1,v+1}} \right) & v = n-1 \\
\varphi_v^{\frac{-\beta(\delta k+k-1)}{k}} \frac{\hat{d}_{nv}\gamma_v}{c_{vv}} & v = n \\
0, & v > n
\end{cases}$$

then, we can write $y_n = \sum_{v=1}^{\infty} h_{nv} X_v$. Here, the necessary and sufficient condition for a series $\sum a_n\gamma_n$ to be summable $\varphi - \mid D, \beta; \delta \mid$ whenever $\sum a_n$ is summable $\varphi - \mid C, \beta; \delta \mid_k$ is

$$\sum |y_n| < \infty \quad \textit{whenever} \quad \sum |X_n|^k < \infty \qquad (6)$$

By Lemma 1, (6) holds if and only if $\sum_{v=1}^{\infty} |\sum_{n=v}^{\infty} h_{nv}u_n|^{k'} < \infty$ whenever $u_n = O(1)$. Then by (2), (3), we can say that $\sum_{v=1}^{\infty} |\sum_{n=v}^{\infty} h_{nv}u_n|^{k'} < \infty$ whenever $u_n = O(1)$ if and only if

$$\left\{ \varphi_v^{\frac{-\beta(\delta k+k-1)}{k}} \left(\frac{d_{vv}}{c_{vv}} |\gamma_v| + \sum_{n=v+1}^{\infty} \left(\frac{\left|\Delta_v\left(\hat{d}_{nv}\gamma_v\right)\right|}{c_{vv}} + \left|\hat{d}_{n,v+1}\gamma_{v+1}\right| \right) + \sum_{n=v+2}^{\infty} \left|\hat{d}_{n,v+1}\gamma_{v+1}\right| \right) \right\} \in l_{k'}$$

holds. This completes the proof of the theorem.

For $\varphi_n = n$ for all values of n, $\beta = 1$ and $\delta = 0$, the necessary and sufficient conditions for $\mid D \mid$ summability of the series $\sum a_n \gamma_n$ whenever the series $\sum a_n$ is summable $\mid C \mid_k$ are obtained.

Data Availability Statement Not applicable.

Declarations

Competing Interests The author has no relevant financial or non-financial interests to disclose.

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