Stability of a point of equilibrium of the mechanical system double planar pendulum: an algebraic view

Abstract

In this work will be presented the deduction of the equations of movement of the double planar pendulum. Immediately after the deduction of these equations, an analysis will be presented on the stability of a point of equilibrium of the system represented by the equations of movement. This analysis will be done using the Theorem of Linearization of Lyapunov-Poincaré and the Criterion of Hurwitz.

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1 Introduction

In this paper, let's make an analysis of the stability of the mechanical system double planar pendulum. Consider a system of n ordinary differential equations of first order nonlinear, with $n \geq 1$ in $\mathbb{Z}$, as it follows

$$\begin{array}{l} \frac{dx_1}{dt} = f_1(x_1, x_2, \ldots, x_n) \\ \frac{dx_2}{dt} = f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ \frac{dx_n}{dt} = f_n(x_1, x_2, \ldots, x_n) \end{array} \qquad (1)$$

Let $x^* = (x_1^*, x_s^*, \ldots, x_n^*)$ be a point which at the same time annul the n equations of this system, i.e.,

$$f_1(x_1^*, x_2^*, \ldots, x_n^*) = 0, \; f_2(x_1^*, x_2^*, \ldots, x_n^*) = 0, \; \ldots, \; f_n(x_1^*, x_2^*, \ldots, x_n^*) = 0.$$

A point under these conditions is called point of equilibrium of the system of differential equations. Moreover, we have that from of a system of n nonlinear first order ordinary differential equations, we can obtain a system where the equations are linear. We note that we achieve this through of a linearization of the nonlinear system around of a point of equilibrium of this system. Thus, consider the linearization of the system (1) around of its point of equilibrium x^*. So, from the linear system we can obtain a matrix A whose elements are the coefficients of this system. The eigenvalues of the matrix A are the roots of the characteristic polynomial, which is obtained by the relation of the determinant $\det(A - \lambda I) = 0$, where I is the identity matrix and A is the matrix of the coefficients of the linearized version of the nonlinear system around of the point of equilibrium of this system and λ_j, $j = 1, \ldots, n$, are the eigenvalues. The matrix of Hurwitz H is constructed from the characteristic polynomial of the matrix A, that is, the elements of this matrix are the coefficients of the characteristic polynomial arranged as follows: in the first line of the matrix, we write the coefficients of the polynomial a_j, $j = 1, \ldots, n$, with odd index, with j increasing; in the second line, we write the coefficients a_j with j even, with j increasing; the other positions are filled with zeros; the next two lines are obtained by moving the first two rows one column to the right, and placing zeros in the positions that have been left empty; this process is followed to construct the other lines, until the independent coefficient of the polynomial occupies the lower right corner. From this matrix we obtain the determinants major minors Δ_j, $j = 1, \ldots, n$. Now, we put a theorem that will be used in the analysis of the stability of the point of equilibrium.

Theorem 1.1 (Criterion of Hurwitz)**.** ([2]) *The eigenvalues of a square matrix A have negative real part if, and only if, all the coefficients of the polynomial characteristic of A are positive and if all the determinants Δ_j, $j = 1, \ldots, n$, obtained of the Hurwitz's matrix are positive.*

Now, consider the problem of to solving (1) with the following initial conditions $x_1(0) = x_1, x_2(0) = x_2, \ldots, x_n(0) = x_n$.

Definition 1.2. Consider the system $\frac{dx}{dt} = f(x)$ (2), where $x \in \mathbb{R}^n$ and $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$. It is denominated flow of the system (2) to function $\Phi : \mathbb{R} \times \Omega \to \mathbb{R}^n$ defined of the following manner: given $y \in \Omega$, consider the solution x of (2) such that $x(0) = y$; given $t \in \mathbb{R}$ we have that $\Phi(t, y) = x(t)$, i.e., for all $y \in \Omega$ fixed the function $t \mapsto \Phi(t, y)$ is the solution of (2) with the initial condition y.

For the problem of solving the system (1) with the initial conditions $x_1, x_2, \ldots, x_n$ the flow is represented by $\Phi(t, \vec{x}(0))$ where $\vec{x}(0) = (x_1, x_2, \ldots, x_n)$ is the initial condition given in the problem. According to [2], the point of equilibrium x^* is asymptotically stable if, and only if, there exists $\delta > 0$ such that: $\|\vec{x}(0) - x^*\| < \delta \Rightarrow \|\Phi(t, \vec{x}(0)) - x^*\| \to 0$ with $t \to \infty$.

Now, we will enunciate the second theorem that will aid in the study of the stability of the point of equilibrium. It should be noted first that for the general case of a function of an open set $\Omega \subseteq \mathbb{R}^n$ in $\mathbb{R}^n$, a linearized system around of a point of equilibrium $\mathbf{O}$ of the nonlinear system can be written by Taylor's expansion as follows $f(x) = Ax + \vartheta(x)$, where A is the matrix of coefficients, i.e., $A_{ij} = \partial f_i(\mathbf{O})/\partial x_j$, which is the Jacobian matrix calculated in the point $\mathbf{O}$ and $\vartheta(x)$ represents the higher order terms in the expansion by the Taylor formula and one can write, $\vartheta(x) = \|x\| \, r(x)$ where $\lim_{x\to 0} r(x) = 0$ or $\lim_{x\to 0} (\vartheta(x)/\|x\|) = 0$.

Theorem 1.3 (Theorem of Linearization of Lyapunov-Poincaré)**.** ([1]) *Let $f(x)$ be a system of nonlinear first-order differential equations continuously differentiable in a neighborhood of a point of equilibrium $\mathbf{O}$ and consider the linearized system around $\mathbf{O}$ obtained by the Taylor series expansion, that is, $f(x) = Ax + \vartheta(x)$. Then, if all the eigenvalues of the square matrix A have negative real part, the equilibrium point $\mathbf{O}$ will be asymptotically stable for the nonlinear system.*

2 The equations of movement of the double planar pendulum

By means of the Lagrange method, we were able to obtain the equations of movement of the planar double pendulum. The Lagrange method can be found in [3]. The situation that we are going to consider is the movement of a double planar pendulum, consisting of two masses, m_1 and m_2, suspended at a fixed point O, which is the origin of the cartesian plane, by means of two rigid and weightless rods of lengths, respectively, l_1 and l_2. Consider only movements of the double pendulum in the vertical plane. According to this situation consider θ and φ the angles formed, respectively, by the stem l_1 with the vertical axis passing through the support point O, and by the stem l_2 with a imaginary axis parallel to the support axis. The mechanical system is composed of two particles, which are represented by the masses of the pendulum, let's say m_1 and m_2. We consider that the orientation of the system is the vertical orientation facing up. Thus, we obtain the Lagrange equations which provide the equations of movement of the mechanical system double planar pendulum: $\ddot{\theta}(t) = \frac{-g}{l_1}\sin(\theta(t)) - \frac{m_2}{m_1+m_2}\frac{l_2}{l_1}(\ddot{\varphi}(t)\cos(\varphi(t)-\theta(t))) + \frac{m_2}{m_1+m_2}\frac{l_2}{l_1}(\dot{\varphi}(t)^2\sin(\varphi(t)-\theta(t)))$ and $\ddot{\varphi}(t) = \frac{-g}{l_2}\sin(\varphi(t)) - \frac{l_1}{l_2}(\ddot{\theta}(t)\cos(\varphi(t)-\theta(t))) - \frac{l_1}{l_2}(\dot{\theta}(t)^2\sin(\varphi(t)-\theta(t)))$, where g is the acceleration of the gravity.

These two equations show how the angles θ and φ vary with the passage of time, i.e., how the pendulum moves at each instant of time t. Thus, we have the equations of movement of the planar double pendulum being a system of two non-linear second order differential equations. So, consider the following physical situation: the double planar pendulum is moving in the middle of any gas and as a result suffers friction caused by the gas. This friction suffered by the pendulum is represented by the so-called dissipative terms, designated by $a\dot{\theta}$ e $b\dot{\varphi}$, where a and b are positive real numbers. With these dissipative terms, we transform these two second order nonlinear differential equations into a nonlinear system of four first order differential equations. It should not be forgotten that the transformed system shows the equations of motion of the planar double pendulum with the dissipative terms of the friction caused by the movement of the pendulum in the middle of a gas. The new nonlinear first order differential equations system that will be used from this moment on is the following:

$\dot{\theta}_1(t) = \theta_2(t)$; $\dot{\theta}_2(t) = \left(\frac{-(m_1+m_2)}{m_1+m_2\sin^2(\varphi_1(t)-\theta_1(t))}\right)\frac{g}{l_1}\sin(\theta_1(t)) - \frac{(m_1+m_2)}{m_1+m_2\sin^2(\varphi_1(t)-\theta_1(t))} .a\theta_2(t) + \left(\frac{m_2}{m_1+m_2\sin^2(\varphi_1(t)-\theta_1(t))}\right)\frac{g}{l_1}\sin(\varphi_1(t))\cos(\varphi_1(t)-\theta_1(t)) + \left(\frac{m_2}{m_1+m_2\sin^2(\varphi_1(t)-\theta_1(t))}\right)\theta_2(t)^2\sin(\varphi_1(t)-\theta_1(t))\cos(\varphi_1(t)-\theta_1(t)) + \left(\frac{m_2}{m_1+m_2\sin^2(\varphi_1(t)-\theta_1(t))}\right)\frac{l_2}{l_1}b\varphi_2(t)\cos(\varphi_1(t)-\theta_1(t)) + \left(\frac{m_2}{m_1+m_2\sin^2(\varphi_1(t)-\theta_1(t))}\right) .\frac{l_2}{l_1}\varphi_2(t)^2\sin(\varphi_1(t)-\theta_1(t))$;

$\dot{\varphi}_1(t) = \varphi_2(t);\ \dot{\varphi}_2(t) = \left(\frac{-(m_1+m_2)}{m_1+m_2\sin^2(\varphi_1(t)-\theta_1(t))}\right)\frac{g}{l_2}\sin(\varphi_1(t))+\frac{(m_1+m_2)}{m_1+m_2\sin^2(\varphi_1(t)-\theta_1(t))}$ $\cdot\frac{g}{l_2}\sin(\theta_1(t))\cos(\varphi_1(t)-\theta_1(t))+\left(\frac{(m_1+m_2)}{m_1+m_2\sin^2(\varphi_1(t)-\theta_1(t))}\right)\frac{l_1}{l_2}a\theta_2(t)\cos(\varphi_1(t)-\theta_1(t))-$ $\left(\frac{-(m_1+m_2)}{m_1+m_2\sin^2(\varphi_1(t)-\theta_1(t))}\right)\frac{l_1}{l_2}\theta_2(t)^2\sin(\varphi_1(t)-\theta_1(t)) - \left(\frac{-(m_1+m_2)}{m_1+m_2\sin^2(\varphi_1(t)-\theta_1(t))}\right)$ $\cdot\frac{l_1}{l_2}b\varphi_2(t) - \left(\frac{m_2}{m_1+m_2\sin^2(\varphi_1(t)-\theta_1(t))}\right)\varphi_2(t)^2\sin(\varphi_1(t)-\theta_1(t))\cos(\varphi_1(t)-\theta_1(t))$, where the definition of the variables is done as follows, $\theta_1(t) \equiv \theta(t);\ \theta_2(t) \equiv \dot{\theta}(t);\ \varphi_1(t) \equiv \varphi(t);\ \varphi_2(t) \equiv \dot{\varphi}(t)$. The objective in the next section will be to carry out the analysis of the stability of the point of equilibrium $(\theta_1(t), \theta_2(t), \varphi_1(t), \varphi_2(t)) = (0,0,0,0)$, which corresponds to the nonlinear system of four first-order differential equations.

3 Stability of the point of equilibrium $(0,0,0,0)$

As mentioned, the objective is to make the stability analysis of the equilibrium point $P = (0,0,0,0)$ of the system of first order differential equations. In this non-linear system, we put $\dot{\theta}_1(t) = f(\theta_1(t), \theta_2(t), \varphi_1(t), \varphi_2(t))$, $\dot{\theta}_2(t) = w(\theta_1(t), \theta_2(t), \varphi_1(t), \varphi_2(t))$, $\dot{\varphi}_1(t) = h(\theta_1(t), \theta_2(t), \varphi_1(t), \varphi_2(t))$ and $\dot{\varphi}_2(t) = p(\theta_1(t), \theta_2(t), \varphi_1(t), \varphi_2(t))$. We expand then these functions in serie of Taylor around of P and for convenience, a new coordinate system is chosen so that the equilibrium point P is translated to the origin (in this case, $P = (0,0,0,0)$ is already at the origin). Moreover, we have that $f(0,0,0,0) = 0$, $w(0,0,0,0) = 0$, $h(0,0,0,0) = 0$ and $p(0,0,0,0) = 0$, since $P = (0,0,0,0)$ is point of equilibrium of the nonlinear system. Thus, the differential equations that govern the evolution of the new variables are, at first approximation, given by another system of four first-order differential equations. Since the functions f, w, h, p were expanded in serie of Taylor and the first approximation in this expansion is taken, the terms obtained are all linear. The new system of equations resulting from this process is as follows: $\bar{\theta}_1(t) = \bar{\theta}_2(t);\ \bar{\theta}_2(t) = \frac{-g}{l_1}\left(\frac{m_1+m_2}{m_1}\right)\bar{\theta}_1(t) - a\left(\frac{m_1+m_2}{m_1}\right)\bar{\theta}_2(t) + \left(\frac{g}{l_1}\frac{m_2}{m_1}\right)\bar{\varphi}_1(t) +$ $\left(b\frac{l_2}{l_1}\frac{m_2}{m_1}\right)\bar{\varphi}_2(t);\ \bar{\varphi}_1(t) = \bar{\varphi}_2(t);\ \bar{\varphi}_2(t) = \frac{g}{l_2}\left(\frac{m_1+m_2}{m_1}\right)\bar{\theta}_1(t) + a\frac{l_1}{l_2}\left(\frac{m_1+m_2}{m_1}\right)\bar{\theta}_2(t) -$ $\frac{g}{l_2}\left(\frac{m_1+m_2}{m_1}\right)\bar{\varphi}_1(t) - b\left(\frac{m_1+m_2}{m_1}\right)\bar{\varphi}_2(t)$.

This new system obtained by linearization is formed by four linear first order differential equations and is called linearized version of the nonlinear system around its point of equilibrium $P = (0,0,0,0)$. Considering this lin-

ear system, we put the coefficients that accompany $\bar{\theta}_1(t)$, $\bar{\theta}_2(t)$, $\bar{\varphi}_1(t)$ and $\bar{\varphi}_2(t)$ in a matrix A, which is the Jacobian matrix calculated in the point of equilibrium $P = (0, 0, 0, 0)$. According to the reciprocal of the Theorem 1.1 the first step for to know if the real part of all eigenvalues of the matrix A is negative, will be to verify how are the coefficients of the polynomial characteristic of that matrix. The characteristic polynomial of the matrix A in the variable λ is given by:
$\lambda^4+\lambda^3\left(\frac{m_1+m_2}{m_1}(a+b)\right)+\lambda^2\left(\frac{m_1^2+m_1m_2}{m_1^2}ab+\frac{m_1+m_2}{m_1}\left(\frac{g}{l_1}+\frac{g}{l_2}\right)\right)+\lambda\left(\frac{m_1^2+m_1m_2}{m_1^2}\left(a\frac{g}{l_2}+b\frac{g}{l_1}\right)\right)+\frac{g^2}{l_1l_2}\left(\frac{m_1^2+m_1m_2}{m_1^2}\right)$.

It is observed that all coefficients of the characteristic polynomial are positive, since m_1 and m_2 are the masses of the pendulum, then positive real numbers, l_1 and l_2 are the lengths of the pendulum wire, then positive real numbers, g the acceleration of gravity is a positive constant and as has been said, a and b which are the dissipative terms of the friction caused by the gas, are positive real numbers. Now, one must examine how are the major minor determinants Δ_1, Δ_2, Δ_3, Δ_4 obtained from the matrix of Hurwitz H. In the case, all these determinants are positive. Thus, by the Theorem 1.1 it follows that all the eigenvalues of the Jacobian matrix A calculated in the point of equilibrium $P = (0, 0, 0, 0)$ of the nonlinear system have real negative part.

It remains to be seen how one can analyze the stability of the equilibrium point P of the original equations, which correspond to the nonlinear system. To do this, let is use the Theorem 1.3. This theorem says that the stability of a point of equilibrium of a nonlinear system is established by the signal of the real part of the eigenvalues λ of the matrix of the coefficients of the linearized system around this point by the expansion in serie of Taylor. Note that the theorem refers to eigenvalues of the matrix of the linear system corresponding to the nonlinear, so it became necessary to linearize the nonlinear system around its point of equilibrium $P = (0, 0, 0, 0)$ so that the matrix A is in the conditions of the Theorem 1.3. Therefore, in the determining of the stability of the point of equilibrium, it is not necessary to explicitly calculate the eigenvalues of the matrix A; it is enough to know the sign of its real parts. Therefore, the Theorem 1.3 provides us which the point of equilibrium P of the nonlinear system is asymptotically stable.

4 Conclusion

It should be noted that this analysis of the stability of the point of equilibrium $P = (0, 0, 0, 0)$ of the nonlinear system was done with the aid of the corresponding linear system. The point P be asymptotically stable for the nonlinear system means that all solutions of this system when in a neighborhood of the point P tend to approaching this point.

It is observed that the result is in agreement with the physical situation, which already foreseen this type of stability for the point of equilibrium.

References

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