CHARACTERIZATIONS OF INVARIANT SUBMANIFOLDS IN LORENTZIAN β-KENMOTSU MANIFOLDS VIA THE TACHIBANA OPERATOR

Abstract. In this study, invariant submanifolds of a Lorentzian β-Kenmotsu manifold have been studied. Invariant submanifolds of Lorentzian β-Kenmotsu manifolds are discussed by using the Tachibana operator. Some important characterizations of Lorentzian β-Kenmotsu manifolds have been obtained by using Tachibana operator under the some special conditions.

1. Introduction

Lorentzian geometry has significant applications, particularly in the fields of general relativity and theoretical physics. Kenmotsu manifolds constitute an important class of contact geometry and stand out in differential geometry research due to their distinct structural properties. β-Kenmotsu manifolds, which are generalizations of Kenmotsu manifolds, offer a broader class characterized by a certain structural function, such as a β-function. This structure provides a more flexible framework compared to classical Kenmotsu manifolds.

The Lorentzian metric introduces a temporal and spatial distinction to manifolds, enabling physical interpretations. Hence, Lorentzian β-Kenmotsu manifolds offer a rich field of study both geometrically and physically. Although various studies have been conducted on Lorentzian Kenmotsu and β-Kenmotsu structures, their submanifolds, curvature properties, and applications remain open to further investigation.

Submanifolds are fundamental structures in differential geometry and have significant applications in various mathematical and physical fields. They serve as crucial tools in understanding the geometric structure of a manifold more effectively. Submanifolds allow for the local analysis of the geometric properties of a larger manifold, which is especially useful for interpreting complex geometric structures. Investigating how special structures on a manifold such as contact structures, complex structures, or Lorentzian metrics project onto submanifolds contributes to the classification and deeper understanding of the ambient manifold.

In physics, the spatial and temporal subdivisions of the universe are often modeled as submanifolds. For instance, the path traced by a particle is a geodesic curve, which is a one-dimensional submanifold. In general relativity, submanifolds are used to represent the distribution of matter and energy within a space-time manifold.

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In surface modeling and geometric processing (e.g., in 3D graphics), submanifold structures play a foundational role. Minimal surfaces, in particular, have important applications in aerodynamics and architecture. Configuration spaces in robotics are often modeled as manifolds, and the possible motions of a robotic arm are described as submanifolds within these spaces.

Characterizing invariant submanifolds of manifolds is an important problem. Invariant submanifolds of $(LCS)_n-$ manifolds by S.K. Hui et al. [1], invariant submanifolds of LP-Sasakian manifolds by V.Venkatesha et al. [2], invariant submanifolds of Kenmotsu manifolds by S.Sular et al. [3], invariant submanifolds of $(k,\mu)-$contact manifolds by M.S. Siddesha et al [4] have been discussed and revealed many important properties of this submanifolds. Similarly, such problems has been addressed by many other authors ([?],[6],[7],[8],[9],[10],[11]). Similarly, S.K. Hui et al. studied the pseudoparallel contact submanifolds of Kenmotsu manifolds in [12] and the Chaki-pseudoparallel invariant submanifolds of Sasakian manifolds in [13].

The Tachibana operator is an important tool in differential geometry, particularly in the field of Riemannian geometry. This operator is used to study various properties of curvature tensors and differential forms. It examines the interaction of curvature tensors, especially with other tensors such as the metric or Ricci tensors. It measures the variation of the curvature structure of a manifold under certain tensors. In other words, the Tachibana operator is a powerful tool for analyzing the variation and symmetry of tensors on manifolds.

Since Lorentz manifolds represent physical space-time models, tensor operations such as the Tachibana operator can be used in conjunction with energy-momentum tensors to produce solutions.

In this study, invariant submanifolds of a Lorentzian β-Kenmotsu Manifold have been studied. Invariant submanifolds of Lorentzian β-Kenmotsu manifolds are discussed by using the Tachibana operator. Some important characterizations of Lorentzian β-Kenmotsu manifolds have been obtained under some special conditions with the help of the Tachibana operator. Given the interesting properties of Lorentzian manifolds and invariant submanifolds, it is evident that the problem addressed in the article is highly significant with respect to the Tachibana operator.

2. Preliminaries

An n-dimensional differentiable manifold $\tilde{\Omega}$ is called Lorentzian β-Kenmotsu manifold if it admits a $(1,1)$-tensor field ϕ, a vector field ξ, a 1-form η and Lorentzian metric g which satisfy the conditions

$$\phi^2\Theta_1 = \Theta_1 + \eta(\Theta_1)\xi,\ g(\Theta_1,\xi) = \eta(\Theta_1), \tag{1}$$

$$\eta(\xi) = -1,\ \phi\xi = 0,\ \eta(\phi\Theta_1) = 0, \tag{2}$$

$$g(\phi\Theta_1,\phi\Theta_2) = g(\Theta_1,\Theta_2) + \eta(\Theta_1)\eta(\Theta_2), \tag{3}$$

for all $\Theta_1,\Theta_2 \in \chi\left(\tilde{\Omega}\right)$, where $\chi\left(\tilde{\Omega}\right)$ is the Lie algebra of smooth vector fields on $\tilde{\Omega}$. Also a Lorentzian β-Kenmotsu manifold $\tilde{\Omega}$ is satisfying

$$\tilde{D}_{\Theta_1}\xi = \beta\left[\Theta_1 - \eta(\Theta_1)\xi\right], \tag{4}$$

$$\left(\tilde{D}_{\Theta_1}\eta\right)(\Theta_2) = \beta\left[g(\Theta_1,\Theta_2) - \eta(\Theta_1)\eta(\Theta_2)\right], \tag{5}$$

$$\left(\tilde{D}_{\Theta_1}\phi\right)(\Theta_2)=\beta\left[g\left(\phi\Theta_1,\Theta_2\right)\xi-\eta\left(\Theta_2\right)\phi\Theta_1\right], \tag{6}$$

where $\tilde{D}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric g. Further, on a Lorentzian β-Kenmotsu manifold $\tilde{\Omega}$ the following relations hold [14, 15]:

$$\tilde{R}\left(\Theta_1,\Theta_2\right)\Theta_3=\beta^2\left[g\left(\Theta_1,\Theta_3\right)\Theta_2-g\left(\Theta_2,\Theta_3\right)\Theta_1\right], \tag{7}$$

$$\tilde{R}\left(\xi,\Theta_2\right)\Theta_3=\beta^2\left[-g\left(\Theta_2,\Theta_3\right)\xi+\eta\left(\Theta_3\right)\Theta_2\right], \tag{8}$$

$$\tilde{R}\left(\Theta_1,\xi\right)\Theta_3=\beta^2\left[g\left(\Theta_1,\Theta_3\right)\xi-\eta\left(\Theta_3\right)\Theta_1\right], \tag{9}$$

$$\tilde{R}\left(\Theta_1,\Theta_2\right)\xi=\beta^2\left[\eta\left(\Theta_2\right)\Theta_1-\eta\left(\Theta_2\right)\Theta_1\right], \tag{10}$$

$$\eta\left(\tilde{R}\left(\Theta_1,\Theta_2\right)\Theta_3\right)=\beta^2 g\left(\eta\left(\Theta_2\right)\Theta_1-\eta\left(\Theta_2\right)\Theta_1,\Theta_3\right), \tag{11}$$

$$g\left(Q\Theta_1,\Theta_2\right)=S\left(\Theta_1,\Theta_2\right)=-\left(n-1\right)\beta^2 g\left(\Theta_1,\Theta_2\right), \tag{12}$$

$$S\left(\Theta_1,\xi\right)=-\left(n-1\right)\beta^2\eta\left(\Theta_1\right), \tag{13}$$

$$S\left(\xi,\xi\right)=\left(n-1\right)\beta^2, \tag{14}$$

$$S\left(\phi\Theta_1,\phi\Theta_2\right)=S\left(\Theta_1,\Theta_2\right)-\left(n-1\right)\beta^2\eta\left(\Theta_1\right)\eta\left(\Theta_2\right), \tag{15}$$

$$Q\Theta_1=-\left(n-1\right)\beta^2\Theta_1,\ Q\xi=-\left(n-1\right)\beta^2\xi, \tag{16}$$

for any vector fields $\Theta_1,\Theta_2,\Theta_3$ on $\tilde{\Omega}$, where $\tilde{R}, S$ and Q denotes the Riemannian curvature tensor, Ricci tensor and Ricci operator on $\tilde{\Omega}$, respectively.

Let Ω be an immersed submanifold of a Lorentzian β-Kenmotsu manifolds Ω. Let $\Gamma\left(T\Omega\right)$ and $\Gamma\left(T^{\perp}\Omega\right)$ be the tangent and normal subspaces of Ω in $\tilde{\Omega}$, respectively. Gauss and Weingarten formulas for $\Gamma\left(T\Omega\right)$ and $\Gamma\left(T^{\perp}\Omega\right)$ are

$$\tilde{D}_{\Theta_1}\Theta_2=D_{\Theta_1}\Theta_2+\sigma\left(\Theta_1,\Theta_2\right), \tag{17}$$

$$\tilde{D}_{\Theta_1}V=-A_V\Theta_1+D^{\perp}_{\Theta_1}V, \tag{18}$$

respectively, for all $\Theta_1,\Theta_2\in\Gamma\left(T\Omega\right)$ and $V\in\Gamma\left(T^{\perp}\Omega\right)$, where D and $D^{\perp}$ are the connections on Ω and $\Gamma\left(T^{\perp}\Omega\right)$, respectively, σ and A are the second fundamental form and the shape operator of Ω. There are related

$$g\left(A_V\Theta_1,\Theta_2\right)=g\left(\sigma\left(\Theta_1,\Theta_2\right),V\right). \tag{19}$$

The covariant derivative of the second fundamental form σ is defined as

$$\left(\tilde{D}_{\Theta_1}\sigma\right)\left(\Theta_2,\Theta_3\right)=D^{\perp}_{\Theta_1}\sigma\left(\Theta_2,\Theta_3\right)-\sigma\left(D_{\Theta_1}\Theta_2,\Theta_3\right)-\sigma\left(\Theta_2,D_{\Theta_1}\Theta_3\right). \tag{20}$$

Specifically, if $\tilde{D}\sigma=0$, Ω is said to be is parallel [5].

Let R be the Riemann curvature tensor of Ω. In this case, the Gauss equation can be expressed as

$$\begin{aligned}\tilde{R}\left(\Theta_1,\Theta_2\right)\Theta_3 &= R\left(\Theta_1,\Theta_2\right)\Theta_3+A_{\sigma\left(\Theta_1,\Theta_3\right)}\Theta_2-A_{\sigma\left(\Theta_2,\Theta_3\right)}\Theta_1\\ &\quad+\left(\tilde{D}_{\Theta_1}\sigma\right)\left(\Theta_2,\Theta_3\right)-\left(\tilde{D}_{\Theta_2}\sigma\right)\left(\Theta_1,\Theta_3\right),\end{aligned} \tag{21}$$

for all $\Theta_1, \Theta_2, \Theta_3 \in \Gamma(T\Omega)$, where if

$$\left(\tilde{D}_{\Theta_1}\sigma\right)(\Theta_2, \Theta_3) - \left(\tilde{D}_{\Theta_2}\sigma\right)(\Theta_1, \Theta_3) = 0,$$

then it is called curvature-invariant submanifold.

Let $\tilde{\Omega}$ be a Riemannian manifold, T is $(0,k)$ −type tensor field and A is $(0,2)$ −type tensor field. In this case, Tachibana tensor field $Q(A,T)$ is defined as

$$\begin{aligned} Q(A,T)(X_1,...,X_k;\Theta_1,\Theta_2) &= -T((\Theta_1 \wedge_A \Theta_2) X_1, ..., X_k) \\ &\quad -... - T(X_1,...,X_{k-1},(\Theta_1 \wedge_A \Theta_2) X_k), \end{aligned} \tag{22}$$

where,

$$(\Theta_1 \wedge_A \Theta_2)\Theta_3 = A(\Theta_2,\Theta_3)\Theta_1 - A(\Theta_1,\Theta_3)\Theta_2, \tag{23}$$

$k \geq 1, X_1, X_2, ..., X_k, \Theta_1, \Theta_2 \in \Gamma\left(T\tilde{\Omega}\right)$.

3. Characterizations of Invariant Submanifolds in Lorentzian β-Kenmotsu Manifolds via the Tachibana Operator

Let Ω be the immersed submanifold of an n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{\Omega}$. If $\phi(T_x\Omega) \subset T_x\Omega$ in every x point, the Ω is called an invariant submanifold. We note that all of properties of an invariant submanifold inherit the ambient manifold. From this section of the article, we will assume that the manifold Ω is the invariant submanifold of the Lorentzian β-Kenmotsu manifold $\tilde{\Omega}$. So, it is clear that

$$\sigma(\Theta_1,\xi) = 0, \sigma(\phi\Theta_1,\Theta_2) = \sigma(\Theta_1,\phi\Theta_2) = \phi\sigma(\Theta_1,\Theta_2) \tag{24}$$

for all $\Theta_1, \Theta_2 \in \Gamma(T\Omega)$.

Moreover, for an invariant submanifold Ω of an n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{\Omega}$, the following relations hold:

$$D_{\Theta_1}\xi = \beta[\Theta_1 - \eta(\Theta_1)\xi], \tag{25}$$

$$R(\Theta_1,\Theta_2)\xi = \beta^2[\eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1], \tag{26}$$

$$R(\xi,\Theta_1)\Theta_2 = \beta^2[-g(\Theta_1,\Theta_2)\xi + \eta(\Theta_2)\Theta_1], \tag{27}$$

$$S(\Theta_1,\xi) = -(n-1)\beta^2\eta(\Theta_1), S(\xi,\xi) = (n-1)\beta^2, \tag{28}$$

$$Q\Theta_1 = -(n-1)\beta^2\Theta_1, Q\xi = -(n-1)\beta^2\xi. \tag{29}$$

Let us examine the $Q(S,\sigma) = 0$ case for the invariant submanifold Ω of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{\Omega}$.

Theorem 1. *Let Ω be an invariant submanifold of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{\Omega}$. If $Q(S,\sigma) = 0$, Ω is a totally geodesic.*

Proof. Let us assume that

$$Q(S,\sigma)(\Theta_4,\Theta_5;\Theta_1,\Theta_2) = 0,$$

for all $\Theta_1, \Theta_2, \Theta_4, \Theta_5 \in \Gamma(T\Omega)$. In this case, we can write

$$\sigma((\Theta_1 \wedge_S \Theta_2)\Theta_4, \Theta_5) + \sigma(\Theta_4, (\Theta_1 \wedge_S \Theta_2)\Theta_5) = 0,$$

and so
(30)
$$\sigma\left(S\left(\Theta_2,\Theta_4\right)\Theta_1-S\left(\Theta_1,\Theta_4\right)\Theta_2,\Theta_5\right)+\sigma\left(\Theta_4,S\left(\Theta_2,\Theta_5\right)\Theta_1-S\left(\Theta_1,\Theta_5\right)\Theta_2\right)=0.$$

If we choose $\Theta_1=\Theta_5=\xi$ in (30) and using $(13),(24)$, we have

$$(n-1)\,\beta^2\sigma\left(\Theta_4,\Theta_2\right)=0.$$

Thus, the proof of the theorem is completed. □

Let us examine the $Q\left(g,\sigma\right)=0$ case for an invariant submanifold Ω of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{\Omega}$.

Theorem 2. *Let Ω be an invariant submanifold of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{\Omega}$. If $Q\left(g,\sigma\right)=0$, Ω is a totally geodesic.*

Proof. Let us assume that

$$Q\left(g,\sigma\right)\left(\Theta_4,\Theta_5;\Theta_1,\Theta_2\right)=0,$$

for all $\Theta_1,\Theta_2,\Theta_4,\Theta_5\in\Gamma\left(T\Omega\right)$. In this case, we can write

$$\sigma\left(\left(\Theta_1\wedge_g\Theta_2\right)\Theta_4,\Theta_5\right)+\sigma\left(\Theta_4,\left(\Theta_1\wedge_g\Theta_2\right)\Theta_5\right)=0,$$

and so
(31)
$$\sigma\left(g\left(\Theta_2,\Theta_4\right)\Theta_1-g\left(\Theta_1,\Theta_4\right)\Theta_2,\Theta_5\right)+\sigma\left(\Theta_4,g\left(\Theta_2,\Theta_5\right)\Theta_1-g\left(\Theta_1,\Theta_5\right)\Theta_2\right)=0.$$

If we choose $\Theta_2=\Theta_4=\xi$ in (31) and taking into account of $(1),(2),(24)$, we obtain

$$\sigma\left(\Theta_1,\Theta_5\right)=0.$$

This proves our assertion. □

Next, we will search the condition $Q\left(g,\tilde{D}\sigma\right)=0$ case for an invariant submanifold Ω of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{\Omega}$.

Theorem 3. *Let Ω be an invariant submanifold of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{\Omega}$. If $Q\left(g,\tilde{D}\sigma\right)=0$, Ω is a totally geodesic.*

Proof. Let us assume that

$$Q\left(g,\tilde{D}\sigma\right)\left(\Theta_4,\Theta_5,\Theta_3;\Theta_1,\Theta_2\right)=0,$$

for all $\Theta_1,\Theta_2,\Theta_3,\Theta_4,\Theta_5\in\Gamma\left(T\Omega\right)$. This implies that

$$\left(\tilde{D}_{\Theta_4}\sigma\right)\left(\left(\Theta_1\wedge_g\Theta_2\right)\Theta_5,\Theta_3\right)+\left(\tilde{D}_{\Theta_4}\sigma\right)\left(\Theta_5,\left(\Theta_1\wedge_g\Theta_2\right)\Theta_3\right)=0,$$

and so
(32)
$$\left(\tilde{D}_{\Theta_4}\sigma\right)\left(g\left(\Theta_2,\Theta_5\right)\Theta_1-g\left(\Theta_1,\Theta_5\right)\Theta_2,\Theta_3\right)+\left(\tilde{D}_{\Theta_4}\sigma\right)\left(\Theta_5,g\left(\Theta_2,\Theta_3\right)\Theta_1-g\left(\Theta_1,\Theta_3\right)\Theta_2\right)=0.$$

If we choose $\Theta_2=\Theta_5=\xi$ in (32) and making use of $(1),(2)$, we have

$$-\left(\tilde{D}_{\Theta_4}\sigma\right)\left(\Theta_1,\Theta_3\right)-\eta\left(\Theta_1\right)\left(\tilde{D}_{\Theta_4}\sigma\right)\left(\xi,\Theta_3\right)+\eta\left(\Theta_3\right)\left(\tilde{D}_{\Theta_4}\sigma\right)\left(\xi,\Theta_1\right)=0. \tag{33}$$

If we use (20) in (33), we get

$$\begin{aligned} & -D^{\perp}_{\Theta_4}\sigma(\Theta_1,\Theta_3)+\sigma(D_{\Theta_4}\Theta_1),\Theta_3+\sigma(\Theta_1,D_{\Theta_4}\Theta_3) \\ & +\beta\eta(\Theta_1)\sigma(\Theta_4,\Theta_3)-\beta\eta(\Theta_3)\sigma(\Theta_4,\Theta_1)=0. \end{aligned} \tag{34}$$

If we choose $\Theta_3=\xi$ in (34) and using (1), (24), we obtain

$$\sigma(\Theta_4,\Theta_1)=0.$$

Thus, the proof is complete. □

Let us examine the $Q\left(S,\tilde{D}\sigma\right)=0$ case for an invariant submanifold Ω of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{\Omega}$.

Theorem 4. *Let Ω be an invariant submanifold of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{\Omega}$. If $Q\left(S,\tilde{D}\sigma\right)=0$, then Ω is a totally geodesic.*

Proof. Let us assume that

$$Q\left(S,\tilde{D}\sigma\right)(\Theta_4,\Theta_5,\Theta_3;\Theta_1,\Theta_2)=0,$$

for all $\Theta_1,\Theta_2,\Theta_3,\Theta_4,\Theta_5\in\Gamma(T\Omega)$. This expansion give us

$$\left(\tilde{D}_{\Theta_4}\sigma\right)((\Theta_1\wedge_S\Theta_2)\Theta_5,\Theta_3)+\left(\tilde{D}_{\Theta_4}\sigma\right)(\Theta_5,(\Theta_1\wedge_S\Theta_2)\Theta_3)=0,$$

and so

$$\begin{aligned} & \left(\tilde{D}_{\Theta_4}\sigma\right)(S(\Theta_2,\Theta_5)\Theta_1-S(\Theta_1,\Theta_5)\Theta_2,\Theta_3) \\ & +\left(\tilde{D}_{\Theta_4}\sigma\right)(\Theta_5,S(\Theta_2,\Theta_3)\Theta_1-S(\Theta_1,\Theta_3)\Theta_2)=0. \end{aligned} \tag{35}$$

If we choose $\Theta_2=\Theta_3=\xi$ in (35) and by means of (28), we have

$$\begin{aligned} & -(n-1)\beta^2\eta(\Theta_5)\left(\tilde{D}_{\Theta_4}\sigma\right)(\Theta_1,\xi)+(n-1)\beta^2\left(\tilde{D}_{\Theta_4}\sigma\right)(\Theta_5,\Theta_1) \\ & +(n-1)\beta^2\eta(\Theta_1)\left(\tilde{D}_{\Theta_4}\sigma\right)(\Theta_5,\xi)=0. \end{aligned} \tag{36}$$

By using (20) in (36), we get

$$\begin{aligned} & (n-1)\beta^3\eta(\Theta_5)\sigma(\Theta_1,\Theta_4)+(n-1)\beta^2\left[D^{\perp}_{\Theta_4}\sigma(\Theta_5,\Theta_1)\right. \\ & \left.-\sigma(D_{\Theta_4}\Theta_5,\Theta_1)-\sigma(\Theta_5,D_{\Theta_4}\Theta_1)\right] \\ & +(n-1)\beta^2\eta(\Theta_1)\left[D^{\perp}_{\Theta_4}\sigma(\Theta_5,\xi)-\sigma(D_{\Theta_4}\Theta_5,\xi)-\sigma(\Theta_5,D_{\Theta_4}\xi)\right]. \end{aligned} \tag{37}$$

If we choose $\Theta_5=\xi$ in (37) and using (1), (24), (28), we obtain

$$-2(n-1)\beta^3\sigma(\Theta_4,\Theta_1)=0.$$

Thus, the proof of the theorem is completed. □

Let us examine the $Q\left(g,\tilde{R}\cdot\sigma\right)=0$ case for an invariant submanifold Ω of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{\Omega}$.

Theorem 5. *Let Ω be an invariant submanifold of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{\Omega}$. If $Q\left(g,\tilde{R}\cdot\sigma\right)=0$, then Ω is a totally geodesic.*

Proof. Let us assume that

$$Q\left(g, \tilde{R}\cdot\sigma\right)(\Theta_4, \Theta_5, \Theta_3, \Theta_6; \Theta_1, \Theta_2) = 0,$$

for all $\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_6 \in \Gamma(T\Omega)$, which implies that

$$\left(\tilde{R}(\Theta_1, \Theta_2)\cdot\sigma\right)((\Theta_4 \wedge_g \Theta_5)\Theta_3, \Theta_6)$$
$$+\left(\tilde{R}(\Theta_1, \Theta_2)\cdot\sigma\right)(\Theta_3, (\Theta_4 \wedge_g \Theta_5)\Theta_6) = 0,$$

that is,

$$\begin{aligned}&\left(\tilde{R}(\Theta_1, \Theta_2)\cdot\sigma\right)(g(\Theta_5, \Theta_3)\Theta_4 - g(\Theta_4, \Theta_3)\Theta_5, \Theta_6)\\ &+\left(\tilde{R}(\Theta_1, \Theta_2)\cdot\sigma\right)(\Theta_3, g(\Theta_5, \Theta_6)\Theta_4 - g(\Theta_4, \Theta_6)\Theta_5) = 0.\end{aligned} \tag{38}$$

If we choose $\Theta_3 = \Theta_4 = \Theta_6 = \xi$ in (38) and using (1), (2), we have

$$\left(\tilde{R}(\Theta_1, \Theta_2)\cdot\sigma\right)(\eta(\Theta_5)\xi + \Theta_5, \xi) = 0. \tag{39}$$

If we use (10) and (24) in (39), we obtain

$$-\beta^2\sigma(\Theta_5, \eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1) = 0. \tag{40}$$

If we choose $\Theta_1 = \xi$ in (40) and by means of (2), (24), we get

$$\sigma(\Theta_5, \Theta_2) = 0.$$

Thus, the proof is completed. □

Now, we will consider the condition $Q\left(S, \tilde{R}\cdot\sigma\right) = 0$ for an invariant submanifold Ω of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{\Omega}$.

Theorem 6. *Let Ω be an invariant submanifold of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{\Omega}$. If $Q\left(S, \tilde{R}\cdot\sigma\right) = 0$, Ω is a totally geodesic.*

Proof. The proof of the theorem can be done in a similar way to the proof of the previous theorem. □

Lemma 1. *On an n-dimensional Lorentzian β-Kenmotsu manifold, the W_1-curvature tensor satisfies the following relations:*

$$W_1(\Theta_1, \Theta_2)\Theta_3 = R(\Theta_1, \Theta_2)\Theta_3 + \frac{1}{(n-1)}\left[S(\Theta_2, \Theta_3)\Theta_1 - S(\Theta_1, \Theta_3)\Theta_2\right], \tag{41}$$

$$W_1(\xi, \Theta_2)\Theta_3 = -\beta^2 g(\Theta_2, \Theta_3)\xi + 2\beta^2\eta(\Theta_3)\Theta_2 + \frac{1}{n-1}S(\Theta_2, \Theta_3)\xi, \tag{42}$$

$$W_1(\Theta_1, \xi)\Theta_3 = \beta^2 g(\Theta_1, \Theta_3)\xi - 2\beta^2\eta(\Theta_3)\Theta_1 - \frac{1}{n-1}S(\Theta_1, \Theta_3)\xi, \tag{43}$$

$$W_1(\Theta_1, \Theta_2)\xi = 2\beta^2\left[\eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1\right], \tag{44}$$

$$\eta(W_1(\Theta_1, \Theta_2)\Theta_3) = 2\beta^2 g(\eta(\Theta_2)\Theta_1 - \eta(\Theta_1)\Theta_2, \Theta_3). \tag{45}$$

Let us examine the $Q(g, W_1\cdot\sigma) = 0$ case for an invariant submanifold Ω of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{\Omega}$.

Theorem 7. *Let Ω be an invariant submanifold of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{\Omega}$. If $Q(g, W_1 \cdot \sigma) = 0$, then Ω is a totally geodesic.*

Proof. We suppose that

$$Q(g, W_1 \cdot \sigma)(\Theta_4, \Theta_5, \Theta_3, \Theta_6; \Theta_1, \Theta_2) = 0,$$

for all $\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_6 \in \Gamma(T\Omega)$. This means that

$$(W_1(\Theta_1, \Theta_2) \cdot \sigma)((\Theta_4 \wedge_g \Theta_5)\Theta_3, \Theta_6)$$
$$+ (W_1(\Theta_1, \Theta_2) \cdot \sigma)(\Theta_3, (\Theta_4 \wedge_g \Theta_5)\Theta_6) = 0,$$

and

$$(W_1(\Theta_1, \Theta_2) \cdot \sigma)(g(\Theta_5, \Theta_3)\Theta_4 - g(\Theta_4, \Theta_3)\Theta_5, \Theta_6)$$
$$+ (W_1(\Theta_1, \Theta_2) \cdot \sigma)(\Theta_3, g(\Theta_5, \Theta_6)\Theta_4 - g(\Theta_4, \Theta_6)\Theta_5) = 0. \tag{46}$$

If we choose $\Theta_3 = \Theta_4 = \Theta_6 = \xi$ in (46) and using (2), we have

$$(W_1(\Theta_1, \Theta_2) \cdot \sigma)(\eta(\Theta_5)\xi + \Theta_5, \xi) = 0. \tag{47}$$

Substituting (24) and (44) in (47), we obtain

$$-2\beta^2 \sigma(\Theta_5, \eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1) = 0. \tag{48}$$

If we choose $\Theta_1 = \xi$ in (48) and by virtue of (2), (24), we get

$$\sigma(\Theta_5, \Theta_2) = 0.$$

Thus, the proof is completed. □

Finally, let us examine the $Q(S, W_1 \cdot \sigma) = 0$ for an invariant submanifold Ω of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{\Omega}$.

Theorem 8. *Let Ω be an invariant submanifold of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{\Omega}$. If $Q(S, W_1 \cdot \sigma) = 0$, then Ω is a totally geodesic.*

Proof. Let us assume that

$$Q(S, W_1 \cdot \sigma)(\Theta_4, \Theta_5, \Theta_3, \Theta_6; \Theta_1, \Theta_2) = 0,$$

for all $\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_6 \in \Gamma(T\Omega)$. In this case, we can write

$$(W_1(\Theta_1, \Theta_2) \cdot \sigma)((\Theta_4 \wedge_S \Theta_5)\Theta_3, \Theta_6)$$
$$+ (W_1(\Theta_1, \Theta_2) \cdot \sigma)(\Theta_3, (\Theta_4 \wedge_S \Theta_5)\Theta_6) = 0,$$

and

$$(W_1(\Theta_1, \Theta_2) \cdot \sigma)(S(\Theta_5, \Theta_3)\Theta_4 - S(\Theta_4, \Theta_3)\Theta_5, \Theta_6)$$
$$+ (W_1(\Theta_1, \Theta_2) \cdot \sigma)(\Theta_3, S(\Theta_5, \Theta_6)\Theta_4 - S(\Theta_4, \Theta_6)\Theta_5) = 0. \tag{49}$$

If we choose $\Theta_1 = \Theta_4 = \Theta_6 = \xi$ in (49), we have

$$(W_1(\xi, \Theta_2) \cdot \sigma)(S(\Theta_5, \Theta_3)\xi, \xi) - S(\xi, \Theta_3)(W_1(\xi, \Theta_2) \cdot \sigma)(\Theta_5, \xi)$$
$$+ S(\Theta_5, \xi)(W_1(\xi, \Theta_2) \cdot \sigma)(\Theta_3, \xi) - S(\xi, \xi)(W_1(\xi, \Theta_2) \cdot \sigma)(\Theta_3, \Theta_5) = 0. \tag{50}$$

Taking $\Theta_3 = \xi$ in (50), we obtain

$$S(\xi, \xi)\sigma(\Theta_5, W_1(\xi, \Theta_2)\xi) = 0. \tag{51}$$

By substituting (28) and (44) into (51) , we have

$$\sigma(\Theta_5, \Theta_2) = 0.$$

Thus, the proof is completed. □

4. Example of Lorentzian β-Kenmotsu Manifolds

Example 1. *Let $\Omega = \left\{ (t_1, t_2, t_3) \in \mathbb{R}^3 \middle| t_3 > 0 \right\}$ be a 3-dimensional manifold. The vector fields*

$$\begin{cases} \xi_1 = e^{t_3} \dfrac{\partial}{\partial t_2}, \\ \xi_2 = e^{t_3} \left(\dfrac{\partial}{\partial t_1} + \dfrac{\partial}{\partial t_2} \right), \\ \xi_3 = \beta \dfrac{\partial}{\partial t_3}, \end{cases}$$

are linearly independent at each point of Ω, where β is a real constant. Let g be the Lorentzian metric defined by

$$\begin{cases} g(\xi_1, \xi_2) = g(\xi_1, \xi_3) = g(\xi_2, \xi_3) = 0, \\ g(\xi_1, \xi_1) = g(\xi_2, \xi_2) = -g(\xi_3, \xi_3) = 1. \end{cases}$$

Let η be the 1-form defined by

$$\eta(X_1) = g(X_1, \xi_3)$$

for any $X_1 \in \chi(\Omega)$ and ϕ be the $(1,1)$-tensor field defined by

$$\phi(\xi_1) = -\xi_2, \phi(\xi_2) = -\xi_1, \phi(\xi_3) = 0.$$

Now using the linearity of ϕ and g, we have

$$\eta(\xi_3) = -1, \phi^2(X_1) = X_1 + \eta(X_1)\xi_3,$$

and

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) + \eta(X_1)\eta(X_2),$$

for all $X_1, X_2 \in \chi(\Omega)$. Let D be the Levi-Civita connection with respect to the Lorentzian metric g. Then we have

$$[\xi_1, \xi_2] = 0, [\xi_1, \xi_3] = -\beta\xi_1, [\xi_2, \xi_3] = -\beta\xi_2.$$

By virtue of Kozsul's formula, we have

$$\begin{array}{lll} D_{\xi_1}\xi_1 = -\beta\xi_3, & D_{\xi_1}\xi_2 = 0, & D_{\xi_1}\xi_3 = -\beta\xi_1, \\ D_{\xi_2}\xi_1 = 0, & D_{\xi_2}\xi_2 = -\beta\xi_3, & D_{\xi_2}\xi_3 = -\beta\xi_2, \\ D_{\xi_3}\xi_1 = 0, & D_{\xi_3}\xi_2 = 0, & D_{\xi_3}\xi_3 = 0. \end{array}$$

Now for

$$X_1 = X_1^1\xi_1 + X_1^2\xi_2 + X_1^3\xi_3$$

and

$$\xi = \xi_3,$$

we have

$$\begin{aligned} D_{X_1}\xi &= D_{X_1^1\xi_1+X_1^2\xi_2+X_1^3\xi_3}\xi_3 \\ &= -\beta\left[X_1^1\xi_1+X_1^2\xi_2\right], \end{aligned} \tag{52}$$

and

$$\beta\left[X_1-\eta\left(X_1\right)\xi\right]=\beta\left[X_1^1\xi_1+X_1^2\xi_2+2X_1^3\xi_3\right], \tag{53}$$

where X_1^1, X_1^2 *and* X_1^3 *are scalars.*
Now using $(52),(53)$, *we have*

$$2\beta\left[X_1^1\xi_1+X_1^2\xi_2+X_1^3\xi_3\right]=0.$$

Since $X_1^1\xi_1+X_1^2\xi_2+X_1^3\xi_3\neq 0$, *therefore we have*

$$\beta=0.$$

Hence it can be easily see that the structure (Ω,ϕ,ξ,η,g) *is a Lorentzian* β*-Kenmotsu manifold* [16].

5. Conclusion

In this study, invariant submanifolds of a Lorentzian β-Kenmotsu Manifold have been studied. Invariant submanifolds of Lorentzian β-Kenmotsu manifolds are discussed by using the Tachibana operator. Some important characterizations of Lorentzian β-Kenmotsu manifolds have been obtained under some special conditions with the help of the Tachibana operator.

Lorentzian geometry has significant applications, particularly in the fields of general relativity and theoretical physics. Submanifolds are fundamental structures in differential geometry and have significant applications in various mathematical and physical fields. The Tachibana operator is an important tool in differential geometry, particularly in the field of Riemannian geometry. This operator is used to study various properties of curvature tensors and differential forms.

Given the interesting properties of Lorentzian manifolds and invariant submanifolds, it is evident that the problem addressed in the article is highly significant with respect to the Tachibana operator.

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