Oscillatory behavior for an enterprise cluster model with multiple delays

Abstract: In this paper, a competition-cooperation enterprise cluster model with delays is studied. The oscillatory behavior of the solutions is investigated. We extend the result in the literature from a mathematical point of view. Make the change of variables and deal with the instability of linearized system. The boundedness of the solutions of the system and the instability of the unique positive equilibrium point will force the system to generate a periodic solutionaround the equilibrium. Two sufficient conditions to guarantee the periodic oscillation of the solutions are provided based on the instability of the equilibrium point, and computer simulations are given to support the present criteria.

Keywords: a cluster model, competition-cooperation, delay, instability, oscillation

AMS Mathematical Subject Classification: 34K11

Original Research Article

1 Introduction

Recently, many researchers have studied various competition-cooperation enterprise models. For example, Liao et al. [1,2] proposed the following delayed differential equation model of the form:

$$\begin{cases} x_1'(t) = r_1 x_1(t)(1 - \frac{x_1(t-\tau_1)}{K_1} - \frac{\alpha(x_2(t-\tau_2)-c_2)^2}{K_2}, \\ x_2'(t) = r_2 x_2(t)(1 - \frac{x_2(t-\tau_1)}{K_2} + \frac{\beta(x_1(t-\tau_2)-c_1)^2}{K_1}), \\ x_1(t) = \psi(t), x_2(t) = \phi(t), t \in [-\max\{\tau_1, \tau_2\}, 0], \end{cases} \tag{1}$$

where the variables $x_1(t)$ and $x_2(t)$ denote the output of two enterprises, respectively; r_1 and r_2 represent the intrinsic growth rates of two enterprises; K_1 and K_2 are the natural market carrying capacity of two enterprises; α is the consumption coefficient of the enterprise with the output $x_2(t)$ to the one with the output $x_1(t)$ and β denotes the transformation coefficient of the enterprise with the output $x_1(t)$ to the one with the output $x_2(t)$. The existence of bifurcation periodic solutions of the model (1) was studied by choosing τ_1 or τ_2 as the bifurcation parameter. Sirghi et al. extended model (1) into distributed delay system [3]. The authors obtained some conditions that the positive equilibrium point of system lossed the stability and induce various oscillations and periodic solutions. Muhammadhaji and Nureji [4] studied the competition and cooperation model of enterprises with variable coefficients in system (1) as follows:

$$\begin{cases} y_1'(t) = y_1(t)[c_1(t) - b_{11}(t)y_1(t-\tau_1) - b_{12}(t)(y_2(t-\tau_2) - d_2(t))^2], \\ y_2'(t) = y_2(t)[c_2(t) - b_{21}(t)y_2(t-\tau_2) - b_{22}(t)(y_1(t-\tau_1) - d_1(t))^2]. \end{cases} \tag{2}$$

A criteria on the periodic solution, extinction, permanence and global attractiveness of the model was obtained by employing the Lyapunov method and the comparison principle. Xu and Shao [5] discussed the periodic solution and global attractivity of model (2) with impulses. Muhammadhaji and Maimaiti [6] studied a non-autonomous competition and cooperation model of two enterprises with discrete feedback controls and constant delays. In [7], the authors considered the following model:

$$\begin{cases} u_1'(t) = u_1(t)[r_1(t) - \alpha_1(t)u_1(t) - \beta_1(t)(u_2(t) - \sigma_2(t))^2 - a_1(t)v_1(t-\eta_1(t))], \\ v_1'(t) = -\delta_1(t)v_1(t) + q_1(t)u_1(t-\zeta_1(t)), \\ u_2'(t) = u_2(t)[r_2(t) - \alpha_2(t)u_2(t) - \beta_2(t)(u_1(t) - \sigma_1(t))^2 - a_2(t)v_2(t-\eta_2(t))], \\ v_2'(t) = -\delta_2(t)v_2(t) + q_2(t)u_2(t-\zeta_2(t)). \end{cases} \tag{3}$$

Some new results on competition and cooperation model of two enterprises with multiple delays and feedback controls were obtained. Lu et al. [8] studied dynamic properties for a discrete competition model with multiple delays and feedback controls. Guerrini discussed the effect of small delays in a competition and cooperation model of enterprises [9]. Xu and Li [10] considered almost periodic solution problems for two enterprises with time-varying delays and feedback controls. Based on periodic time scales theory and the fixed point theorem of strict-set-contraction, Peng et al. discussed a class of enterprise cluster models with feedback controls and time-varying delays on time scales. Some new sufficient conditions for the existence of positive periodic solutions are obtained [11]. Ren et al. discussed the role of gradient learning on the convergence of output dynamics in the duopoly competition model under incomplete information [12,13]. For a model of competition and cooperation between two enterprises with reaction, diffusion, and delays, the stability and Hopf bifurcation for variants with delays were considered by examining a system of delay equations analytically and numerically [14, 15]. In [16], the authors investigated the following delayed competitive-cooperative systems:

$$\begin{cases} z_1'(t) = z_1(t)[r_1(t) - a_{11}^1(t)z_1(t-\tau) - a_{12}^1(t)z_1(t-2\tau) - a_{12}(t)z_2(t-2\tau) + a_{13}(t)z_3(t-\tau)], \\ z_2'(t) = z_2(t)[r_2(t) - a_{21}(t)z_1(t-2\tau) - a_{22}^1(t)z_2(t-\tau) - a_{22}^2(t)z_2(t-2\tau) + a_{23}(t)z_3(t-\tau)], \\ z_3'(t) = z_3(t)[r_3(t) + a_{31}(t)z_1(t-\tau) + a_{32}(t)z_2(t-\tau) - a_{33}^1(t)z_3(t) - a_{33}^2(t)z_3(t-\tau)]. \end{cases} \tag{4}$$

Some sufficient conditions to ensure the permanent and globally attractive of the system (4) were provided. Guo et al. discussed a three-dimensional competition system in shifting environments, and the existence of forced waves for diffusive competition systems was derived [17-20]. For other effects of cluster systems, one can see [21-22]. In [23], the authors investigated the following competition-cooperation enterprise cluster model with one core enterprise and two satellite enterprises:

$$\begin{cases} x_1'(t) = x_1(t)[r_1 - a_1x_1(t) - b_1x_2(t) - c_1(y(t-\tau) - d)^2)], \\ x_2'(t) = x_2(t)[r_2 - a_2x_2(t) - b_2x_1(t) - c_1(y(t-\tau) - d)^2)], \\ y'(t) = y(t)[r_3 - a_3y(t) + c_2((x_1(t-\tau) - d_1)^2 + (x_2(t-\tau) - d_2)^2)]. \end{cases} \tag{5}$$

The upper bounds of both core enterprise and satellite enterprise outputs were provided. By selecting τ as the bifurcating parameter, the conditions of local stability and Hopf bifurcation were discussed. In this paper, we extend models (1) and (6) to the following competition-cooperation enterprise cluster model with two core enterprises and two satellite enterprises and multiple delays system:

$$\begin{cases} x_1'(t) = x_1(t)[r_1 - a_1x_1(t) - b_1x_2(t-\tau_2) - c_1(y_1(t-\tau_3) - d_3)^2 - c_2(y_2(t-\tau_4) - d_4)^2)], \\ x_2'(t) = x_2(t)[r_2 - a_2x_2(t) - b_2x_1(t-\tau_1) - c_2(y_1(t-\tau_3) - d_3)^2 - c_1(y_2(t-\tau_4) - d_4)^2)], \\ y_1'(t) = y_1(t)[r_3 - a_3y_1(t) - b_3y_2(t-\tau_4) + c_3(x_1(t-\tau_1) - d_1)^2 + c_4(x_2(t-\tau_2) - d_2)^2)], \\ y_2'(t) = y_2(t)[r_4 - a_4y_2(t) - b_4y_1(t-\tau_3) + c_4(x_1(t-\tau_1) - d_1)^2 + c_3(x_2(t-\tau_2) - d_2)^2)], \\ x_i(t) = \phi_i(t), y_i(t) = \psi_i(t), t \in [-\max\{\tau_1, \cdots, \tau_4\}, 0], \end{cases} \tag{6}$$

where all the parameter values are positive real numbers. $x_i(t), y_i(t) (i = 1, 2)$ are the outputs of satellite enterprises and core enterprises, respectively, r_i are the intrinsic growth rates, a_i are the self-regulations of enterprises, b_i are the competition rates of enterprises, c_1 and c_2 are the competition rates between satellite enterprises and core enterprises, c_3

and c_4 are the rates of conversion of commodity into the reproduction of enterprises, d_i are initial outputs of enterprises. Our goal is to investigate the periodic oscillation of the system (6). We believe that the general bifurcating method is a tool to deal with this system. However, it is pointed out that if time delays are different from real numbers, the bifurcating method is not easy to determine the existence of bifurcation periodic solution due to the complexity of bifurcation equations. In the present paper, we will use the mathematical analysis method to study the existence of oscillatory solutions of the system (6).

2 Preliminaries

Obviously, $(0,0,0,0)^T$ is a trivial equilibrium point of the system (6). However, we shall be concerned with the non-trivial positive equilibrium point. Since $r_i(i=1,\cdots,4)$ are positive real numbers, there exists a positive equilibrium point of system (6), say $(x_1^*, x_2^*, y_1^*, y_2^*)^T$. Then make the change of variables $x_i(t) \to x_i(t) - x_i^*, y_i(t) \to y_i(t) - y_i^*, (i=1,2)$, noting that

$$\begin{cases} r_1 - a_1 x_1^* - b_1 x_2^* - c_1(y_1^* - d_3)^2 - c_2(y_2^* - d_4)^2 = 0, \\ r_2 - a_2 x_2^* - b_2 x_1^* - c_2(y_1^* - d_3)^2 - c_1(y_2^* - d_4)^2 = 0, \\ r_3 - a_3 y_1^* - b_3 y_2^* + c_3(x_1^* - d_1)^2 + c_4(x_2^* - d_2)^2 = 0, \\ r_4 - a_4 y_2^* - b_4 y_1^* + c_4(x_1^* - d_1)^2 + c_3(x_2^* - d_2)^2 = 0, \end{cases} \tag{7}$$

We have

$$
\begin{cases}
x_1'(t) = -a_1x_1^*x_1(t) - b_1x_1^*x_2(t-\tau_2) - 2c_1x_1^*(y_1^*-d_3)y_1(t-\tau_3) - 2c_2x_1^*(y_2^*-d_4)y_2(t-\tau_4) \\
-a_1x_1^2(t) - b_1x_1(t)x_2(t) - c_1x_1(t)y_1^2(t-\tau_3) - 2c_1(y_1^*-d_3)x_1(t)y_1(t-\tau_3) \\
-c_2x_1(t)y_2^2(t-\tau_4) - 2c_2(y_2^*-d_4)x_1(t)y_2(t-\tau_4) - c_1x_1^*y_1^2(t-\tau_3) - c_2x_1^*y_2^2(t-\tau_4), \\
x_2'(t) = -a_2x_2^*x_2(t) - b_2x_2^*x_1(t-\tau_1) - 2c_2x_2^*(y_1^*-d_3)y_1(t-\tau_3) - 2c_1x_2^*(y_2^*-d_4)y_2(t-\tau_4) \\
-a_2x_2^2(t) - b_2x_1(t-\tau_1)x_2(t) - c_2x_2(t)y_1^2(t-\tau_3) - 2c_2(y_1^*-d_3)x_2(t)y_1(t-\tau_3) \\
-c_1x_2(t)y_2^2(t-\tau_4) - 2c_1(y_2^*-d_4)x_2(t)y_2(t-\tau_4) - c_2x_2^*y_1^2(t-\tau_3) - c_1x_2^*y_2^2(t-\tau_4), \\
y_1'(t) = -a_3y_1^*y_1(t) - b_3y_1^*y_2(t-\tau_4) + 2c_3y_1^*(x_1^*-d_1)x_1(t-\tau_1) + 2c_4y_1^*(x_2^*-d_2)x_2(t-\tau_2) \\
-a_3y_1^2(t) - b_3y_1(t)y_2(t-\tau_4) + c_3y_1(t)x_1^2(t-\tau_1) + c_4y_1(t)x_2^2(t-\tau_2) \\
+2c_3(x_1^*-d_1)y_1(t)x_1(t-\tau_4) + 2c_4(x_2^*-d_2)y_1(t)x_2(t-\tau_2) + c_3y_1^*x_1^2(t-\tau_1) + c_4y_1^*x_2^2(t-\tau_2), \\
y_2'(t) = -a_4y_2^*y_2(t) - b_4y_2^*y_1(t-\tau_1) + 2c_4y_2^*(x_1^*-d_1)x_1(t-\tau_1) + 2c_3y_2^*(x_2^*-d_2)x_2(t-\tau_2) \\
-a_4y_2^2(t) - b_4y_2(t)y_1(t-\tau_3) + c_4y_2(t)x_1^2(t-\tau_1) + c_3y_2(t)x_2^2(t-\tau_2) \\
+2c_4(x_1^*-d_1)y_2(t)x_1(t-\tau_4) + 2c_3(x_2^*-d_2)y_2(t)x_2(t-\tau_2) + c_4y_2^*x_1^2(t-\tau_1) + c_3y_2^*x_2^2(t-\tau_2),
\end{cases} \tag{8}
$$

System (8) can be expressed in the following matrix form:

$$
u'(t) = Au(t) + Bu(t-\tau) + f(u(t), u(t-\tau)), \tag{9}
$$

where $u(t) = [x_1(t), x_2(t), y_1(t), y_2(t)]^T$, $u(t-\tau) = [x_1(t-\tau_1), x_2(t-\tau_2), y_1(t-\tau_4), y_2(t-\tau_4)]^T$, A and B both are 4×4 matrices, and $f(u(t), u(t-\tau))$ is a four by one vector: $f(u(t), u(t-\tau)) = [-a_1x_1^2(t) - b_1x_1(t)x_2(t) - c_1x_1(t)y_1^2(t-\tau_3)\cdots - c_1x_1^*y_1^2(t-\tau_3) - c_2x_1^*y_2^2(t-\tau_4), \cdots - a_4y_2^2(t) - b_4y_2(t)y_1(t-\tau_3)\cdots c_4y_2^*x_1^2(t-\tau_1) + c_3y_2^*x_2^2(t-\tau_2)]^T$,

$$
A = diag(a_{11}, a_{22}, a_{33}, a_{44}) = diag(-a_1x_1^*, -a_2x_2^*, -a_3y_1^*, -a_4y_2^*),
$$

$$
B = (b_{ij})_{4\times 4} = \begin{pmatrix} 0 & b_{12} & b_{13} & b_{13} \\ b_{21} & 0 & b_{23} & b_{24} \\ b_{31} & b_{32} & 0 & b_{34} \\ b_{41} & b_{42} & b_{43} & 0 \end{pmatrix},
$$

where $b_{12} = -b_1x_1^*, b_{13} = -2c_1x_1^*(y_1^*-d_3), b_{14} = -2c_2x_1^*(y_2^*-d_4), b_{21} = -b_2x_2^*, b_{23} = -2c_2x_2^*(y_1^*-d_3), b_{24} = -2c_1x_2^*(y_2^*-d_4), b_{31} = 2c_3y_1^*(x_1^*-d_1), b_{32} = 2c_4y_1^*(x_2^*-d_2), b_{34} = -b_3y_1^*, b_{41} = 2c_4y_2^*(x_1^*-d_1), b_{42} = 2c_3y_2^*(x_2^*-d_2), b_{43} = -b_4y_2^*$. The linearized system of (9) is

$$
u'(t) = Au(t) + Bu(t-\tau) \tag{10}
$$

Lemma 1 If matrix $M = A + B$ is a nonsingular matrix for selected parameters, then there exists a unique positive equilibrium point for system (6).

Proof To prove that there exists a unique positive equilibrium point for system (6), we only need to prove that system (10) has a unique trivial equilibrium point. Assume that $u^* = (x_1^*, x_2^*, y_1^*, y_2^*)^T$ is an equilibrium point of the system (10), then we have the following algebraic equations:

$$(A + B)u^* = Mu^* = \mathbf{0} \tag{11}$$

According to the basic linear algebraic knowledge, system (11) has a unique solution since $M = A + B$ is a nonsingular matrix. This unique solution exactly is the trivial solution. Noting that $f(\mathbf{0}) = \mathbf{0}$. Therefore, system (8) (or (9)) has a unique trivial solution, implying that system (6) has a unique positive equilibrium point.

Lemma 2 All solutions of system (6) are bounded.

Proof Noting that all parameter values are positive real number. Thus, from (6) we have

$$\begin{cases} x_1'(t) \le x_1(t)[r_1 - a_1 x_1(t) - b_1 x_2(t - \tau_2)], \\ x_2'(t) \le x_2(t)[r_2 - a_2 x_2(t) - b_2 x_1(t - \tau_1)]. \end{cases} \tag{12}$$

To prove the boundedness of the positive solutions in the system (12), consider a Lyapunov function $V(t) = \frac{1}{2}(x_1^2(t) + x_2^2(t))$. Then we get

$$\begin{aligned} V(t)'|_{(12)} &= x_1(t)x_1'(t) + x_2(t)x_2'(t) \\ &\le -a_1 x_1^3(t) - a_2 x_2^3(t) - b_1 x_1^2(t) x_2(t - \tau_2) \\ &\quad -b_2 x_2^2(t) x_1(t - \tau_1) + r_1 x_1^2(t) + r_2 x_2^2(t) \end{aligned} \tag{13}$$

Noting that $x_1^3(t), x_2^3(t)$ are higher order infinity than $x_1^2(t), x_2^2(t)$ as $x_1 \to \infty, x_2 \to \infty$. Since $a_i > 0, b_i > 0 (i = 1, 2)$, so, there exists suitably large $N > 0$ such that $V(t)'|_{(12)} < 0$ as $x_1(t) > N$ and $x_2(t) > N$. This means that $x_1(t), x_2(t)$ are bounded, say $x_1(t) \le M_1, x_2(t) \le M_2$. From (6) we have

$$\begin{cases} y_1'(t) \le y_1(t)[(r_3 + c_3 M_1^2 + c_4 M_2^2) - a_3 y_1(t) - b_3 y_2(t - \tau_4)], \\ y_2'(t) \le y_2(t)[(r_4 + c_4 M_1^2 + c_3 M_2^2) - a_4 y_2(t) - b_4 y_1(t - \tau_3)]. \end{cases} \tag{14}$$

By constructing a Lyapunov function $V(t) = \frac{1}{2}(y_1^2(t) + y_2^2(t))$, one can prove that both of $y_1(t)$ and $y_2(t)$ are bounded in system (14). Thus, all solutions of the system (6) are bounded.

3 The existence of periodic oscillatory solutions

Theorem 1 Assume that the system (6) has a unique positive equilibrium point. Let $\beta_1, \beta_2, \beta_2, \beta_4$ be characteristic values of matrix B. If there is some β_i, say β_1, such that $Re(\beta_1) > |a_{11}| = a_1 x_1^*$, then the unique positive equilibrium point of system (6) is unstable, implying that there exists an oscillatory solution in system (6).

Proof Noting that the nonlinear term $f(u(t), u(t-\tau))$ of the system (9) is a higher order infinitesimal as $u_i \to u_i^*$. Obviously, if the trivial solution of system (10) is unstable, then the unique positive equilibrium point of system (6) is also unstable. Therefore, to discuss the instability of the positive equilibrium point of the system (6), we only need to deal with the instability of the trivial solution of the system (10). Since $\beta_1, \beta_2, \beta_2, \beta_4$ are characteristic values of matrix B, then the characteristic equations corresponding to the system (10) are the following:

$$\Pi_{i=1}^{4}(\lambda - a_{ii} - \beta_i e^{-\lambda\tau_i}) = 0. \tag{15}$$

Thus, we are led to investigate the nature of the roots for the following equation

$$\lambda - a_{11} - \beta_1 e^{-\lambda\tau_1} = 0. \tag{16}$$

Noting that equation (16) is a transcendental equation, it is hard to find all solutions for the equation. However, we show that there exists a positive real part eigenvalue of the equation (16) under the assumptions of Theorem 1. Indeed, if $Re(\beta_1) > |a_{11}| = a_1 x_1^*$, setting $\lambda = \sigma + i\omega, \beta_1 = \beta_{11} + i\beta_{12}, \sigma = Re(\lambda), \omega = Im(\lambda), \beta_{11} = Re(\beta_1), \beta_{12} = Im(\beta_1)$. Separating the real part and imaginary part of the equation (16) we get

$$\sigma = a_{11} + \beta_{11} e^{-\sigma\tau_1}\cos(\omega\tau_1) - \beta_{12} e^{-\sigma\tau_1}\sin(\omega\tau_1) \tag{17}$$

We show that the equation (17) has a positive root. Let

$$\phi(\sigma) = \sigma - a_{11} - \beta_{11} e^{-\sigma\tau_1}\cos(\omega\tau_1) + \beta_{12} e^{-\sigma\tau_1}\sin(\omega\tau_1) \tag{18}$$

Obviously, $\phi(\sigma)$ is a continuous function of σ. Noting that $\beta_{11} > |a_{11}|$, then $\phi(0) = -a_{11} - \beta_{11}\cos(\omega\tau_1) + \beta_{12}\sin(\omega\tau_1) \leq -a_{11} - \beta_{11} < 0$. as $\omega\tau_1 \sim 2n\pi$, where n is an integer number. Since $\lim_{\sigma\to+\infty} e^{-\sigma\tau_1} = 0$, so there exists a suitably large σ, say $\sigma_1 (> 0)$ such that $\phi(\sigma_1) = \sigma_1 - a_{11} - \beta_{11} e^{-\sigma_1\tau_1}\cos(\omega\tau_1) + \beta_{12} e^{-\sigma_1\tau_1}\sin(\omega\tau_1) > 0$. By the Intermediate Value Theorem, there exists a σ, say $\sigma_0 \in (0, \sigma_1)$ such that $\phi(\sigma_0) = 0$, implying that there is a positive real part characteristic value of equation (16). This means that the trivial solution of system (10) is unstable, implying that the trivial solution of system (8) is unstable. It suggests that the unique positive equilibrium point $(x_1^*, x_2^*, y_1^*, y_2^*)^T$ of system (6) is unstable. This instability of the unique positive equilibrium point together with the boundedness of the solutions will force system (6) to generate a limit circle, namely, there is a periodic oscillatory solution [24, 25]. The proof is completed.

Now set $a = \min_{1\le i\le 4} |a_{ii}|, b = \max_j \sum_{i=1}^4 |b_{ij}|$. Then we have

Theorem 2 Assume that the conditions of Lemma 1 and Lemma 2 hold. If the following inequality holds:

$$-a + b > 0. \tag{19}$$

Then the unique positive equilibrium point of system (6) is unstable, implying that system (6) generates a periodic oscillatory solution.

Proof Noting that $|v(t)| = v(t)$ as $v(t) > 0$, and $|v(t)| = -v(t)$ as $v(t) < 0$. So from (10) we have

$$\begin{cases} \frac{d|x_1(t)|}{dt} \le -a_1x_1^*|x_1(t)| + |b_1x_1^*||x_2(t-\tau_2)| + |2c_1x_1^*(y_1^* - d_3)||y_1(t-\tau_3)| \\ \qquad +|2c_2x_1^*(y_2^* - d_4)||y_2(t-\tau_4)|, \\ \frac{d|x_2(t)|}{dt} \le -a_2x_2^*|x_2(t)| + |b_2x_2^*||x_1(t-\tau_1)| + |2c_2x_2^*(y_1^* - d_3)||y_1(t-\tau_3)| \\ \qquad +|2c_1x_2^*(y_2^* - d_4)||y_2(t-\tau_4)|, \\ \frac{d|y_1(t)|}{dt} \le -a_3y_1^*|y_1(t)| + |b_3y_1^*||y_2(t-\tau_4)| + |2c_3y_1^*(x_1^* - d_1)||x_1(t-\tau_1)| \\ \qquad +|2c_4y_1^*(x_2^* - d_2)||x_2(t-\tau_2)|, \\ \frac{d|y_2(t)|}{dt} \le -a_4y_2^*|y_2(t)| + |b_4y_2^*||y_1(t-\tau_1)| + |2c_4y_2^*(x_1^* - d_1)||x_1(t-\tau_1)| \\ \qquad +|2c_3y_2^*(x_2^* - d_2)||x_2(t-\tau_2)|. \end{cases} \tag{20}$$

Let $z(t) = \sum_{i=1}^2 (|x_i(t)| + |y_i(t)|)$. From (20) we have

$$\frac{dz(t)}{dt} \le -az(t) + bz(t-\tau) \tag{21}$$

Consider a scalar delayed differential equation

$$\frac{dw(t)}{dt} = -aw(t) + bw(t-\tau) \tag{22}$$

Obviously, we have $z(t) \le w(t)$. We prove that the trivial solution of (22) is unstable. Indeed, the characteristic equation associated with the equation (22) is the following

$$\lambda = -a + be^{-\lambda\tau} \tag{23}$$

Similar to Theorem 1, consider a function $\psi(\lambda) = \lambda + a - be^{-\lambda\tau}$. Then $\psi(0) = a - b < 0$ since $-a + b > 0$. There exists a suitably large λ, say $\lambda_1(> 0)$ such that $\psi(\lambda_1) =$

$\lambda_1 + a - be^{-\lambda_1 \tau} > 0$. So there exists a λ, say $\lambda_0 \in (0, \lambda_1)$ such that $\psi(\lambda_0) = 0$, implying that there is a positive real part characteristic value of equation (22). This means that the trivial solution of system (10) is unstable. It suggests that the unique positive equilibrium point $(x_1^*, x_2^*, y_1^*, y_2^*)^T$ of system (6) is unstable, and system (6) has an oscillatory solution.

4 Simulation result

This simulation is based on the system (6). Firstly, the parameters are selected as the following: $r_1 = 0.78, r_2 = 0.86, r_3 = 0.82, r_4 = 0.84, a_1 = 0.75, a_2 = 0.76, a_3 = 0.94, a_4 = 0.98, b_1 = 0.65, b_2 = 0.64, b_3 = 0.58, b_4 = 0.52, c_1 = 0.12, c_2 = 0.05, c_3 = 3.85, c_4 = 3.65, d_1 = 0.25, d_2 = 0.28, d_3 = 0.22, d_4 = 0.24$. Then the unique positive equilibrium point is $(0.2328, 0.7504, 0.9912, 1.2011)^T$. Therefore, matrix A=diag$(-0.1746, -0.5703, -0.9315, -1.1772)$, $b_{12} = -0.1513, b_{13} = -0.0431, b_{14} = -0.0223, b_{21} = -0.4803, b_{23} = -0.0578, b_{24} = -0.1728, b_{31} = -0.1526, b_{32} = 1.4273, b_{34} = -0.5749, b_{41} = -0.1754, b_{42} = 4.3368, b_{43} = -0.6243$. The characteristic values of matrix B are $0.3211 \pm 0.1265i$, $-0.3211 \pm 0.7617i$. Noting that $0.3211 > 0.2328$, the conditions of Theorem 1 are satisfied. When time delays are selected as $\tau_1 = 2.75, \tau_2 = 2.82, \tau_3 = 2.78, \tau_4 = 2.72$, and $\tau_1 = 3.15, \tau_2 = 3.22, \tau_3 = 3.18, \tau_4 = 3.12$, respectively, there exists an oscillatory solution for the system (6) (see Fig.1). Then we select parameter values as $r_1 = 0.85, r_2 = 0.86, r_3 = 0.70, r_4 = 0.72, a_1 = 0.85, a_2 = 0.86, a_3 = 0.98, a_4 = 0.92, b_1 = 0.65, b_2 = 0.56, b_3 = 0.68, b_4 = 0.52, c_1 = 0.24, c_2 = 0.046, c_3 = 3.78, c_4 = 3.72, d_1 = 0.45, d_2 = 0.16, d_3 = 0.42, d_4 = 0.84$. Thus, the unique positive equilibrium point is $(0.3294, 0.6639, 0.7018, 1.4882)^T$. Therefore, matrix A=diag$(-0.2801, -0.5711, -0.6878, -1.3691)$, $b_{12} = -0.2141, b_{13} = -0.0446, b_{14} = -0.0185, b_{21} = -0.3696, b_{23} = -0.0171, b_{24} = -0.1981, b_{31} = -0.6441, b_{32} = 3.7944, b_{34} = -0.4828, b_{41} = -1.2928, b_{42} = 5.4734, b_{43} = -0.7531$. Therefore, $a = 0.2801, b = 9.0537$, and $-0.2801 + 9.0537 > 0$. The conditions of Theorem 2 are satisfied. When time delays are selected as $\tau_1 = 4.35, \tau_2 = 4.25, \tau_3 = 4.38, \tau_4 = 4.15$, and $\tau_1 = 4.75, \tau_2 = 4.65, \tau_3 = 4.78, \tau_4 = 4.55$, respectively, there exists an oscillatory solution for the system (6) (see Fig.2). In figure 3, we only change the values of $c_i (i = 1, \cdots, 4)$, the other parameter values are the same as in figure 2. Finaly, the parameter values are selected as $r_1 = 0.84, r_2 = 0.76, r_3 = 0.80, r_4 = 0.82, a_1 = 0.85, a_2 = 0.86, a_3 = 0.98, a_4 = 0.92, b_1 = 0.45, b_2 = 0.36, b_3 = 0.48, b_4 = 0.32, c_1 =$

$0.12, c_2 = 0.08,\ c_3 = 3.25, c_4 = 2.65, d_1 = 0.15, d_2 = 0.18, d_3 = 0.12, d_4 = 0.14$. Then the unique positive equilibrium point is $(0.5616, 0.4155, 0.9133, 1.2586)^T$. Therefore, matrix A=diag$(-0.4774, -0.3573, -0.8951, -1.1579)$ $b_{12} = -0.2527, b_{13} = -0.1078, b_{14} = -0.1792, b_{21} = -0.1496, b_{23} = -0.0532,\ b_{24} = -0.1117, b_{31} = 2.6714, b_{32} = 1.1617, b_{34} = -0.4384, b_{41} = 2.8016, b_{42} = 1.9656, b_{43} = -0.4028$. The characteristic values of matrix B are $0.3877, 0.1956, -0.2916 \pm 1.0182i$. Noting that $0.3877 > 0.3573$, $-a = -0.3573, b = 5.4730$, and $-a + b = -0.3573 + 5.4730 > 0$. Both of the conditions of Theorem 1 and Theorem 2 are satisfied. When time delays are selected as $\tau_1 = 2.75, \tau_2 = 2.65, \tau_3 = 2.78, \tau_4 = 2.62$, and $\tau_1 = 3.25, \tau_2 = 3.15, \tau_3 = 3.28, \tau_4 = 3.12$, respectively, there exists an oscillatory solution for the system (6) (see Fig.4).

5 Conclusion

In this paper, we have discussed the oscillatory behavior of the solutions for a cluster model with two core enterprises and two satellite enterprises. Based on the method of mathematical analysis, we provided two theorems to guarantee the oscillation of the solutions. Some simulations are provided to indicate the effectiveness of the criteria. From the simulation we see that the oscillatory frequency of the two core enterprises and the two satellite enterprises almost the same, respectively, The amplitude of the core enterprises is greater than the satellite enterprises. It is tally with the actual situations.

Competing Interests

The author has declared that no competing interests exist.

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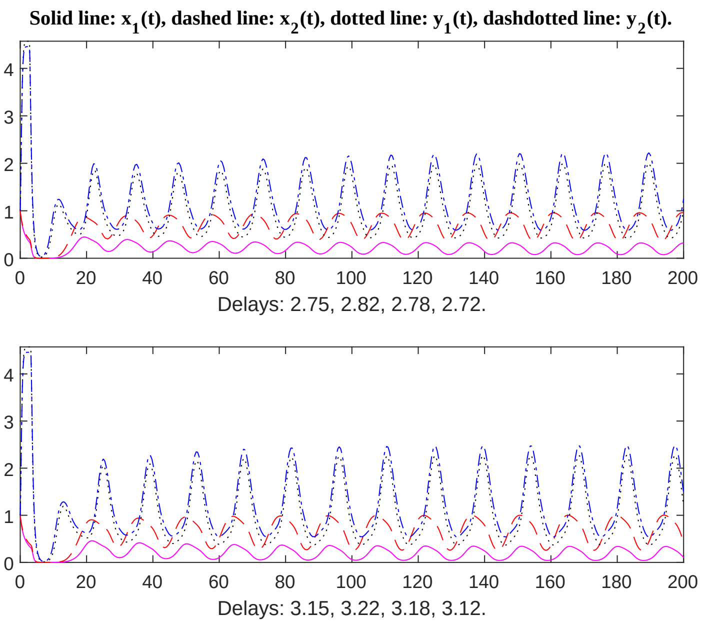
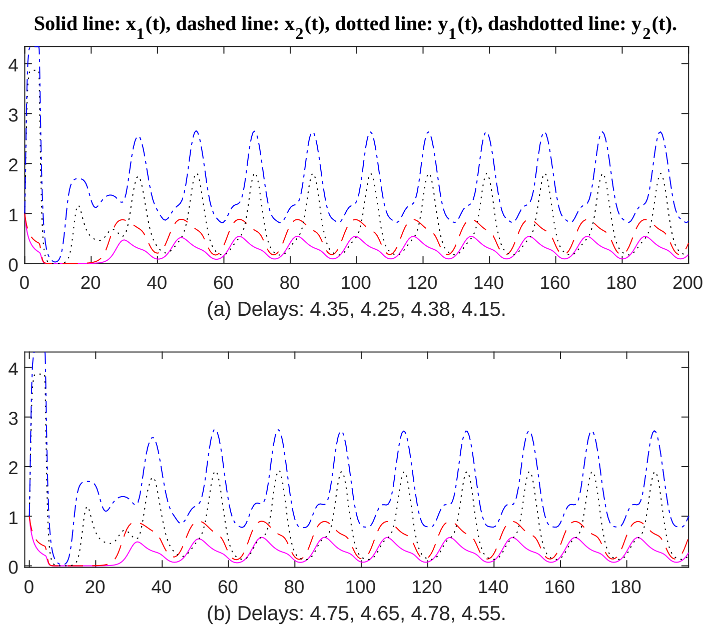
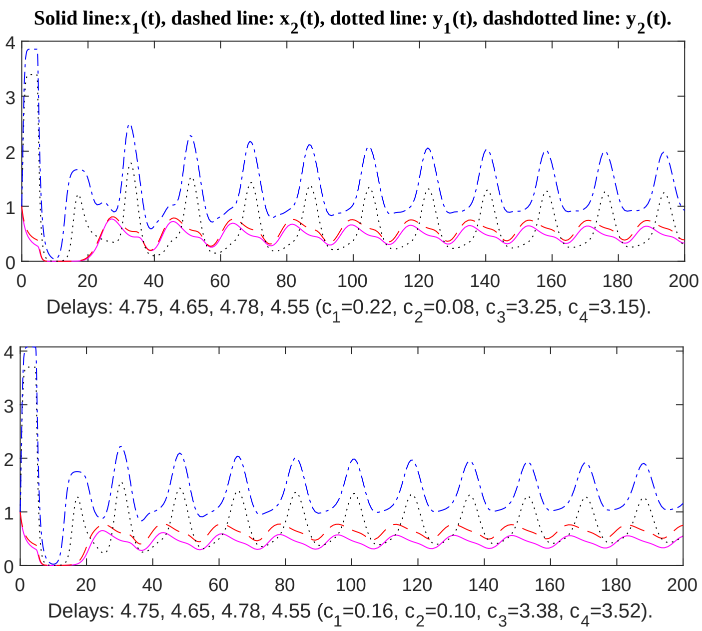


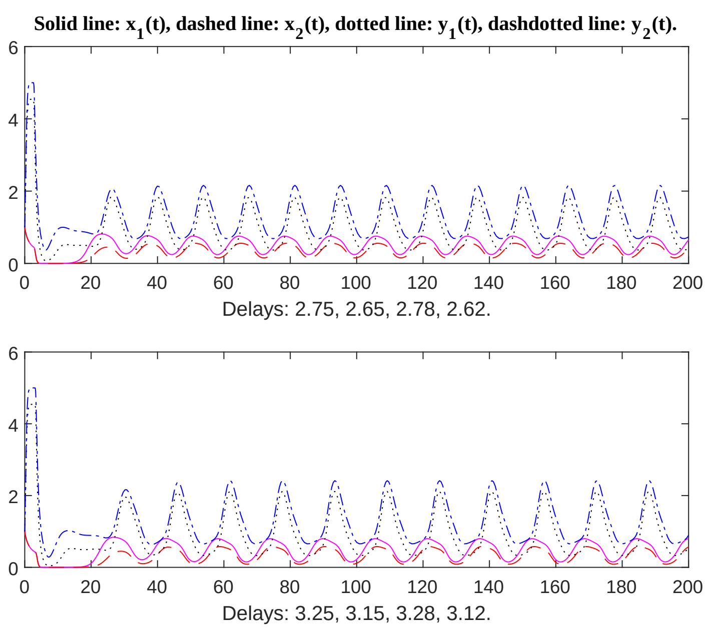
(a) Fig. 1. Oscillation of the solutions.



(b) Fig.2. Oscillation of the solutions.



(a) Fig.3. Oscillation of the solutions.



(b) Fig.4. Oscillation of the solutions.