**An Investigation of the Velocity Profile in Poiseuille Flow of an Incompressible Fluid between Concentric Circular Cylinders**

**Abstract**

The aim of this study is to obtain an analytical solution for the Poiseuille flow of an incompressible fluid confined between two fixed concentric circular cylinders with radii ​ and​, under slip boundary conditions and certain simplifying assumptions. Two distinct cases have been considered: parallel flow with slip boundary conditions on both cylinder surfaces, and combinations of slip and no-slip conditions on the inner and outer surfaces, respectively. For each scenario, the steady-state velocity profile and the volume flow rate have been derived and analyzed under various boundary conditions. The findings demonstrate that both the velocity distribution and flow rate are significantly influenced by the annular gap between the cylinders and the slip length. Additionally, it was observed that as the radius of the gap increases, the fluid velocity distribution decreases, with the maximum velocity shifting towards the centerline of the flow in the vertical motion.

Keywards: Fluid velocity; incompressible fluid; poiseuille flow; circular cylinder.

**1 Introduction**

Fluid mechanics is a field of science that studies how fluids behave both when they are at rest and when they are in motion. It therefore covers the static, kinematic, and dynamic properties of fluids (Bansal, 2005). Incompressible flow refers to a type of fluid motion where the flow speed is much lower than the speed of sound in the fluid, making changes in density negligible. With this definition, most of the fluids and flow situations we encounter in everyday life can be considered incompressible. Scientists study incompressible flows through theoretical analysis, experiments, and computer simulations. As modern flow systems become more compact and efficient, the demands on computational tools for simulating incompressible flows have grown significantly. Just as accurate prediction is vital for aerodynamic performance, it is now essential for incompressible flow simulations as well. This has led to the development of advanced methods and software for modeling incompressible flows, particularly with the help of high-performance computing resources.

From a mathematical perspective, modeling incompressible flow presents distinct challenges that do not arise in compressible flow equations, mainly due to the condition that the fluid’s density must remain constant. Physically, in an incompressible fluid, information is assumed to propagate instantaneously, which places strict demands on numerical methods to maintain this condition and complicates the design of boundary conditions at the outlet. The different approaches to solving incompressible flow equations mainly stem from the various techniques used to enforce the incompressibility constraint. The Navier–Stokes (N–S) equations, which are partial differential equations, describe the motion of a viscous, incompressible fluid. These equations are among the most fundamental and widely used models in computational fluid dynamics, applicable to both liquids and gases under a range of flow conditions (Dylan et al., 2012). In general, there are two main methods for solving the equations that govern viscous incompressible flows, known as the incompressible Navier–Stokes equations. The first method enforces the incompressibility condition directly. The second method treats the problem using a compressible flow framework, where the momentum and continuity equations are linked by incorporating the fluid’s density (Victor, 1962). The major difference between the incompressible and compressible N-S formulations is in the continuity equation (Dochan et al., 2010).

**2 Poiseuille Flow of Circular Pipe**

Poiseuille flow refers to the fully developed, steady, laminar motion of a viscous, incompressible fluid through a straight pipe with a constant circular cross-section. In this flow, the fluid’s velocity is directly proportional to the pressure difference and the fourth power of the pipe’s radius, while it is inversely proportional to the pipe’s length and the fluid’s viscosity. This classical solution remains fundamental in applied viscous fluid dynamics research.

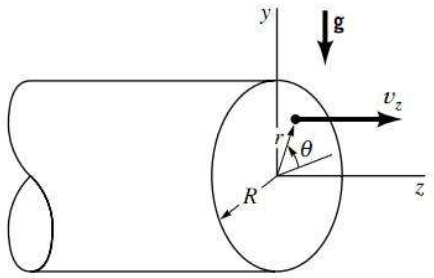
In a cylindrical polar coordinate system with the Z-axis aligned along the pipe’s axis, the problem’s axial symmetry makes the angular direction irrelevant. Letting R represent the pipe’s radius, the no-slip condition applies: ,  ………………….. (1)

, ,  ……… (2)

The continuity equation is automatically satisfied, as the momentum equation which reduces to:

 …………..….. (3)

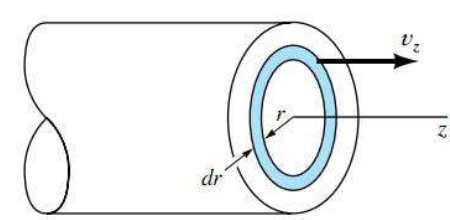
As for the flow between plates, both sides must be equal to the same constant *G* (say). With the boundary conditions (1) and the restriction that the velocity be finite at the tube axis for unsteady flow of motion (Laglois and Deville, 2014). Poiseuille flow is pressure induced flow in a long duct usually a pipe. Specifically, it is assumed that there is laminar flow of an incompressible Newtonian fluid of viscosity  induced by a constant positive pressure difference or pressure drop  in a pipe of length and radius . By a pipe is meant a right circular cylindrical duct that is a duct with a circular cross section normal to its axis or generator. Geometrically, Poiseuille flow is analyzed using cylindrical polar coordinates with origin on the center line of the pipe entrance and z-direction aligned with the centerline (Fig. 1).



**Fig. 1. Coordinate system for Poiseuilli flow**

The axial velocity profile is parabolic (Fig. 2):

 ……………….. (4)



**Fig. 2. Flow through differential annular ring**

The volume rate of flow through the pipe is 

If the pipe is rotate about its own axis, not necessary with constant angular velocity. The boundary conditions (1) must be modified.

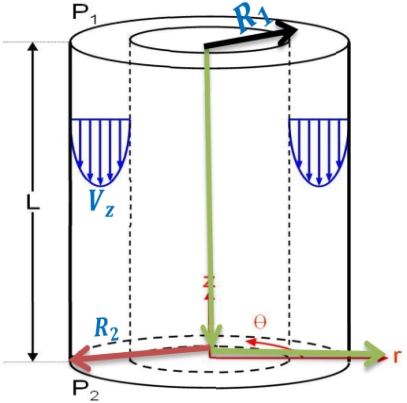
,  ………..….. (6)

,  …….……… (7)

If  where  is the Poiseuille solutions in equation (4).

In this study Poisseuille flow occurs in the gap between two fixed parallel flows to the walls of concentric circular cylinders is untouched title. The inner cylinder of radius  is at rest and the outer cylinder of radius  is stationary. The apparatus has a height L which is much larger than the radius of either cylinder so that the apparatus height is supposed finite. The cylindrical coordinates system in the steady state velocity field is such that the (velocity field) is then determined from the integration of the momentum equation:





**Fig. 3. Poiseuille flow between concentric cylinders**

The motion of fluid contained between two concentric circular pipes of constant radius  and with  is as indicated in fig. 3. The pipes rotate about their common axis with constant angular velocities and respectively. In addition the pipes may translate steadily, parallel to their common axis; suppose the outer pipe stationary with velocity *U* relative to the inner. Define a cylindrical coordinate system  which fixed with the inner but does not rotate with it. The z-axis lies along the common axis of the pipes, because of the axial symmetry, the orientation of the  axis is unimportant. Since the cylindrical coordinate system is not accelerated, the fluid motion is governed by the continuity equation and momentum equation. The no-slip condition requires that:

,   ……….……….. (9)

and    …………..….. (10)

Analytical solutions for both Newtonian and inelastic non-Newtonian fluids with slip boundary conditions in Couette and Poiseuille flows using the Navier linear and non-linear slip laws (Ferras et al., 2012). The solutions are included with different slip laws or different slip coefficients at different walls. Without assumption, it would not be possible to obtain a simple solution for Navier-Stoke equations (McDonough, 2009). The analytical solution of Couette-Poiseuille flow of Bingham fluids between two porous parallel plates with slip conditions was found (Chen and Zhu, 2008). On the other hand, the analytical solutions of some fully developed flows of couple stress fluid between concentric cylinders with slip boundary condition found (Devakar et al., 2014). Song and Chen (2008) investigated the Poiseuille flow of simple fluids in cylindrical nanochannels. Hron et al., (2008) established closed form analytical solution for the flows of incompressible non-Newtonian fluids with Navier’s slip conditions at the boundary. Three distinct cases had been identified in Couette flow of incompressible fluid between concentric rotating cylinders ( Temesgen Dagu et al., 2016). Due to its vast applications in engineering and industry, pressure driven flow or the Poiseuille flow has attracted attention of various researchers. Although a pressure driven flow are unidirectional and have been studied earlier for Newtonian and no-Newtonian fluids, they still attract special attention in a number of emerging problems (Tang, 2012). Many authors tried to investigate closed analytical solutions of Newtonian and no-Newtonian fluid with slip and no slip condition on horizontal plan Poiseuille flow, channel with different dimensional geometry and cross section with linear and non-linear slip laws. But this are not consider an incompressible fluid between vertical concentric circular cylinders governed by a Navier-Stokes equation with three dimensional geometry. Poiseuille flow occurs in the gap between two fixed parallel flows to the walls of concentric circular cylinders subject to slip boundary conditions by applying some appropriate assumptions with constants pressure gradient. The aim of this study is to construct analytical solutions of a Poiseulle flow of an incompressible fluid between concentric circular cylinders by applying some appropriate assumptions.

**3 Methodology**

This research was carried out using a combination of analytical methods and experimental investigations. The overall approach followed a standard procedure, which involved several systematic steps to address each problem effectively. Reasonable assumptions are made to simplification the parallel flow to the wall with the slip on boundary condition at ;   and  steady flow, fully developed laminar and Newtonian with constant density, viscosity and pressure gradient those used in Poiseuille flow where  is represents the slip length and is radius and  is length of a cylinders. In this research, the governing equations for mass and momentum conservation were formulated and simplified based on standard assumptions. These simplified equations were then integrated to derive expressions for key variables, including velocity and volumetric flow rate, as applicable to Poiseuille flow. The resulting integration constants were determined using appropriate boundary conditions. Analytical solutions for the velocity profile and flow rate were developed and visualized using MATLAB for various numerical values.

**4 Results**

The fluid is expected to travel parallel to the wall between two fixed concentric cylinders with slip at boundary condition only the z-components exist and verified are given below.

Flow conditions: No change in the axial or in z-direction is assumed.

In r-direction, in r-directionfunction of, in -direction, then the equation of continuity:  …………………. (11)

Since the fluid is Newtonian and incompressible and density is constant thenand . Thus the equation (11) becomes. Sois independent of distance from the inlet and that the velocity profile  and  are appear the same for all value of z.

The momentum equation in radial component:

……… (12)

The momentum equation in tangential component:

………. (13)

The momentum equation in axial component:

 …………………. (14)

Based on the above assumptions the axial component of the momentum equation is reduced



or  ……………………………. (15)

In which total derivatives are used because dependes only on , from the assumptions it is clear that equation (15) can be denoted , since where p is a constant pressure gradient.

In which both sides of the equation are two times integration of equation (15):

)

or  …………………. (16)

Again integration equation (16) with respective to r

 ……………………… (17)

where A and B are unknown constants.

**CASE I:** Consider two fixed long concentric circular cylinders with the inner cylinder radius  and the outer cylinder radius. Determine the steady state velocity distribution and the volume rate of flow distribution in the field. The boundary conditions are that the fluid does not slip at the surface of the inner cylinder and the fluid slip at the surface of the outer cylinder. When the outer cylinder and the inner cylinder are fixed, then the boundary conditions are:  at  and  at 

The two constants may be evaluated by apply the following boundary condition of zero velocity at the surface of the inner cylinder and the non-zero velocity at the surface of outer cylinder. To determine A and B are both slip boundary condition and from equation (17).

no slip,  at  then  .....………….. (18)

slip,  at  then  ……………. (19)

Solving equations (18) and (19) to get

 and 

Substitution the values of A and B in equation (17) to get the final expression of the velocity profile:

………… (20)

where  is a constant or velocity specified and k is slip length or friction coefficient.



**Fig. 4. Velocity profile**

Fig. 4 illustrates that as  and  for , the fluid velocity increases. The figure also indicates that the maximum velocity occurs along the centerline of the parallel flow, considering a no-slip condition at the inner surface and a slip condition at the outer surface of the concentric cylinder, particularly when . Conversely, the fluid velocity decreases near the boundaries where the inner surface has a slip condition and the outer surface has a no-slip condition, specifically when or. The graphs further show that reducing the gap between the concentric cylinders leads to a decrease in the fluid velocity.

A reduction in the slip length at the boundary condition in the parallel flow causes a decrease in fluid velocity. In other words, when the slip and no-slip conditions at the surfaces of the concentric cylinder are closer together, the influence of the slip length on the velocity becomes less significant.

**CASE II:** Consider two stationary, long concentric circular cylinders, where the inner cylinder has a radius and the outer cylinder has a radius​. The objective is to determine the steady-state velocity profile and the corresponding volumetric flow rate within the annular region. The boundary conditions specify that the fluid exhibits slip at the inner cylinder surface, while there is no slip at the outer cylinder surface. When the outer cylinder and the inner cylinder are fixed, then the boundary conditions are:

 at  and  at 

The two constants may be evaluated by apply the boundary condition of non-zero velocity at the inner cylinder and the zero velocity at the outer cylinder. To determine A and B are both slip boundary condition and from equation (17).

slip,  at  then  .....….…….. (21)

no slip,  at  then  ……………. (22)

Solving equations (21) and (22) to get

 and 

Substitution the values of A and B in equation (17) to get the final expression of the velocity profile:

………… (23)

where  is a constant or velocity specified and k is slip length or friction coefficient.



**Fig. 5. Velocity profile**

Fig. 5 shows that as as  and  for , the fluid velocity increases. It is also evident that the maximum velocity occurs along the centerline of the parallel flow, where the inner surface allows slip and the outer surface enforces a no-slip condition, particularly when. On the other hand, the fluid velocity decreases near the slip boundary at the inner surface and the no-slip boundary at the outer surface of the concentric cylinder, specifically when the radius or.

It is also observed that reducing the gap between the concentric cylinders lowers the fluid velocity. Similarly, a shorter slip length at the boundary condition in parallel flow decreases the velocity. In other words, the closer the slip and no-slip boundaries are on the surfaces of the concentric cylinder, the less influence the slip length has on the fluid velocity.

**5 Conclusion**

This study considered two cases to derive analytical solutions for Poiseuille flow of an incompressible fluid between two fixed vertical concentric cylinders with slip boundary conditions. In each case, the fluid velocity was examined for various slip lengths, and the effect of the radial coordinate r was analyzed. The results show that the velocity field depends only on the axial component and varies with both r and the slip length k. The velocity reaches its maximum at the centerline and decreases toward the slip boundary on the inner surface and the no-slip boundary on the outer surface. A shorter slip length reduces the velocity, and when the slip and no-slip surfaces are closer, the slip length has less influence. Overall, the analytical solutions, based on the given assumptions and slip conditions, provide clear velocity profiles for different slip lengths and radial positions within the two concentric cylinders.

**References**

Bansal, R.K. (2005). A Text Book of Fluid Mechanics and Hydraulic Machines, *Laxmi Publication Ltd.,* *113, Golden House, Daryaganj, New Delhi-110002*.

Chen,Y-L. and Zhu, K.Q. (2008). Couette-Poiseuille flow of Bingham fluid between two porous parallel plates with slip conditions, *Journal of Non-Newtonian Fluid Mechanics*, 153(1):1-11.

Devakar, M., Sreenivasu, D., and Shankar, B. (2014). Analytical solutions of some fully developed flows of couple stress fluid between concentric cylinders with slip boundary conditions, *Int.J.Eng.Math.,*1-13.

Dochan, K., Cetin, K. and Chang, S.K. (2010). Computational Challenges of viscous Incompressible Flow, *NASA-Ames Research Center, Moffet Field*.

Dylan, N., Minot, A., Mocz, P. and Liu, P. (2012). AM274 Final Project: Continuous Galerkin Navier-Stokes in 2D, 1.

Ferrras, L.L., Oberga, M.N. and Pinho, F.T. (2012). Analytical solutions for Newtonian and inelastic non-Newtonian flows with wall slip, *Journal of Non-Newtonian Fluid Mechanics*, 175:76-88.

Hron, j., Le-Roux, C., Malek, J. and Rajagopal, K.R.(2008). Flows of incompressible fluids Subject to Navier's slipon the boundary, *Computers and Mathematics with Applications*, 56(8):2128-2143.

Laglois,W.E , and Deville, M.O. (2014). Slow viscous flow, 116-126, [www.springer.com/978-3-319-03834-6](http://www.springer.com/978-3-319-03834-6)

McDonough, J.M. (2009). Lectures in Elementary Fluid Dynamics, 122-126.

Song, X. and Chen, J.K. (2008). A comparative study on Poiseuille flow of simple fluids through cylindrical and slit-like nanochannels, *International Journal of Heat and Mass Transfer*, 51(7-8):177 0-1779.

Temesgen, D., Daba M., and Hailu, K. (2016). Analysis of acircularcouette flow of impressible fluid over acircular cylinder.(unpublished).

Victor, L.S. (1962). Fluid Mechanics, *McGraw-Hill Book Company, Inc*, 174-177.