SOME IMPORTANT CHARACTERIZATIONS ACCORDING TO THE TACHIBANA OPERATOR FOR INVARIANT SUBMANIFOLDS OF A LORENTZIAN β-KENMOTSU MANIFOLD

Abstract. In this study, invariant submanifolds of a Lorentzian β-Kenmotsu Manifold have been studied. Invariant submanifolds of Lorentzian β-Kenmotsu manifolds are discussed by using the Tachibana operator. Some important characterizations of Lorentzian β-Kenmotsu manifolds have been obtained under some special conditions with the help of the Tachibian operator.

1. Introduction

Lorentzian geometry has significant applications, particularly in the fields of general relativity and theoretical physics. Kenmotsu manifolds constitute an important class of contact geometry and stand out in differential geometry research due to their distinct structural properties. β-Kenmotsu manifolds, which are generalizations of Kenmotsu manifolds, offer a broader class characterized by a certain structural function, such as a β-function. This structure provides a more flexible framework compared to classical Kenmotsu manifolds.

The Lorentzian metric introduces a temporal and spatial distinction to manifolds, enabling physical interpretations. Hence, Lorentzian β-Kenmotsu manifolds offer a rich field of study both geometrically and physically. Although various studies have been conducted on Lorentzian Kenmotsu and β-Kenmotsu structures, their submanifolds, curvature properties, and applications remain open to further investigation.

Submanifolds are fundamental structures in differential geometry and have significant applications in various mathematical and physical fields. They serve as crucial tools in understanding the geometric structure of a manifold more effectively. Submanifolds allow for the local analysis of the geometric properties of a larger manifold, which is especially useful for interpreting complex geometric structures. Investigating how special structures on a manifold such as contact structures, complex structures, or Lorentzian metrics project onto submanifolds contributes to the classification and deeper understanding of the ambient manifold.

In physics, the spatial and temporal subdivisions of the universe are often modeled as submanifolds. For instance, the path traced by a particle is a geodesic curve, which is a one-dimensional submanifold. In general relativity, submanifolds are used to represent the distribution of matter and energy within a space-time manifold.

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In surface modeling and geometric processing (e.g., in 3D graphics), submanifold structures play a foundational role. Minimal surfaces, in particular, have important applications in aerodynamics and architecture. Configuration spaces in robotics are often modeled as manifolds, and the possible motions of a robotic arm are described as submanifolds within these spaces.

Characterizing invariant submanifolds of manifolds is an important problem. Invariant submanifolds of $(LCS)_n-$ manifolds by S.K. Hui et al. [1], invariant submanifolds of LP-Sasakian manifolds by V.Venkatesha et al. [2], invariant submanifolds of Kenmotsu manifolds by S.Sular et al. [3], invariant submanifolds of $(k,\mu)-$contact manifolds by M.S. Siddesha et al [4] have been discussed and revealed many important properties of this submanifolds. Similarly, this problem has been addressed by many other authors ([5],[6],[7],[8],[9]). Similarly, S.K. Hui et al. studied the pseudoparallel contact submanifolds of Kenmotsu manifolds in [10] and the Chaki-pseudoparallel invariant submanifolds of Sasakian manifolds in [11].

In this paper we investigated invariant submanifolds of Lorentzian Kenmotsu manifolds. We obtained some important characterizations for total geodesic submanifolds of Lorentzian Kenmotsu manifolds. We considered pseudoparallel, 2-pseudoparallel, Ricci generalized pseudoparallel, Ricci generalized 2-pseudoparallel submanifolds of these manifolds one by one and studied the geometry of Lorentzian Kenmotsu manifolds.

2. Preliminaries

A n-dimensional differentiable manifold $\tilde{M}$ is called Lorentzian β-Kenmotsu manifold if it admits a $(1,1)$-tensor field ϕ, a vector field ξ, a 1-form η and Lorentzian metric g which satisfy the conditions

$$\phi^2 X = X + \eta(X)\xi,\ g(X,\xi) = \eta(X), \tag{1}$$

$$\eta(\xi) = -1,\ \phi\xi = 0,\ \eta(\phi X) = 0, \tag{2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{3}$$

for all $X, Y \in \chi\left(\tilde{M}\right)$, where $\chi\left(\tilde{M}\right)$ is the Lie algebra of smooth vector fields on $\tilde{M}$. Also a Lorentzian β-Kenmotsu manifold $\tilde{M}$ is satisfying

$$\tilde{\nabla}_X \xi = \beta\left[X - \eta(X)\xi\right], \tag{4}$$

$$\left(\tilde{\nabla}_X \eta\right)(Y) = \beta\left[g(X,Y) - \eta(X)\eta(Y)\right], \tag{5}$$

$$\left(\tilde{\nabla}_X \phi\right)(Y) = \beta\left[g(\phi X, Y)\xi - \eta(Y)\phi X\right], \tag{6}$$

where $\tilde{\nabla}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric g. Further, on a Lorentzian β-Kenmotsu manifold $\tilde{M}$ the following relations hold [13, 14]:

$$\tilde{R}(X,Y)Z = \beta^2\left[g(X,Z)Y - g(Y,Z)X\right], \tag{7}$$

$$\tilde{R}(\xi,Y)Z = \beta^2\left[-g(Y,Z)\xi + \eta(Z)Y\right], \tag{8}$$

$$\tilde{R}(X,\xi)Z = \beta^2\left[g(X,Z)\xi - \eta(Z)X\right], \tag{9}$$

$$\tilde{R}(X,Y)\xi = \beta^2 [\eta(Y)X - \eta(Y)X], \tag{10}$$

$$\eta\left(\tilde{R}(X,Y)Z\right) = \beta^2 g(\eta(Y)X - \eta(Y)X, Z), \tag{11}$$

$$g(QX,Y) = S(X,Y) = -(n-1)\beta^2 g(X,Y), \tag{12}$$

$$S(X,\xi) = -(n-1)\beta^2 \eta(X), \tag{13}$$

$$S(\xi,\xi) = (n-1)\beta^2, \tag{14}$$

$$S(\phi X,\phi Y) = S(X,Y) - (n-1)\beta^2 \eta(X)\eta(Y), \tag{15}$$

$$QX = -(n-1)\beta^2 X, \ \ Q\xi = -(n-1)\beta^2 \xi, \tag{16}$$

for any vector fields X, Y, Z on $\tilde{M}$, where $\tilde{R}, S$ and Q denotes the curvature tebsor, Ricci tensor and Ricci operator on $\tilde{M}$.

Let M be the immersed submanifold of a Lorentzian β-Kenmotsu manifolds M. Let the tangent and normal subspaces of M in $\tilde{M}$ be $\Gamma(TM)$ and $\Gamma\left(T^{\perp}M\right)$, respectively. Gauss and Weingarten formulas for $\Gamma(TM)$ and $\Gamma\left(T^{\perp}M\right)$ are

$$\tilde{\bigtriangledown}_X Y = \bigtriangledown_X Y + \sigma(X,Y), \tag{17}$$

$$\tilde{\bigtriangledown}_X V = -A_V X + \bigtriangledown^{\perp}_X V, \tag{18}$$

respectively, for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma\left(T^{\perp}M\right)$, where $\bigtriangledown$ and $\bigtriangledown^{\perp}$ are the connections on M and $\Gamma\left(T^{\perp}M\right)$, respectively, σ and A are the second fundamental form and the shape operator of M. There is a relation

$$g(A_V X, Y) = g(\sigma(X,Y), V) \tag{19}$$

between the second basic form and shape operator defined as above. The covariant derivative of the second fundamental form σ is defined as

$$\left(\tilde{\bigtriangledown}_X \sigma\right)(Y,Z) = \bigtriangledown^{\perp}_X \sigma(Y,Z) - \sigma(\bigtriangledown_X Y, Z) - \sigma(Y, \bigtriangledown_X Z). \tag{20}$$

Specifically, if $\tilde{\bigtriangledown}\sigma = 0$, M is said to be is parallel second fundamental form [5].

Let R be the Riemann curvature tensor of M. In this case, the Gauss equation can be expressed as

$$\begin{aligned} \tilde{R}(X,Y)Z &= R(X,Y)Z + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X \\ &+ \left(\tilde{\bigtriangledown}_X \sigma\right)(Y,Z) - \left(\tilde{\bigtriangledown}_Y \sigma\right)(X,Z), \end{aligned} \tag{21}$$

for all $X, Y, Z \in \Gamma(TM)$, where if

$$\left(\tilde{\nabla}_X \sigma\right)(Y,Z) - \left(\tilde{\nabla}_Y \sigma\right)(X,Z) = 0,$$

then it is called curvature-invariant submanifold.

Let $\tilde{M}$ be a Riemannian manifold, T is $(0,k)$ −type tensor field and A is $(0,2)$ −type tensor field. In this case, Tachibana tensor field $Q(A,T)$ is defined as

$$\begin{aligned} Q(A,T)(X_1,...,X_k;X,Y) &= -T((X \wedge_A Y)X_1,...,X_k) \\ &-... - T(X_1,...,X_{k-1},(X \wedge_A Y)X_k), \end{aligned} \tag{22}$$

where,

$$(X \wedge_A Y) Z = A(Y,Z) X - A(X,Z) Y, \tag{23}$$

$k \geq 1, X_1, X_2, ..., X_k, X, Y \in \Gamma\left(T\tilde{M}\right)$.

3. Characterizations of Lorentzian β-Kenmotsu Manifolds According to the Tachibana Operator

Let M be the immersed submanifold of an n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{M}$. If $\phi(T_x M) \subset T_x M$ in every x point, the M manifold is called invariant submanifold. We note that all of properties of an invariant submanifold inherit the ambient manifold. From this section of the article, we will assume that the manifold M is the invariant submanifold of the Lorentzian β-Kenmotsu manifold $\tilde{M}$. So, it is clear that

$$\sigma(X,\xi) = 0, \sigma(\phi X, Y) = \sigma(X, \phi Y) = \phi\sigma(X,Y) \tag{24}$$

for all $X, Y \in \Gamma(TM)$.

Moreover, for an invariant submanifold M of an n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{M}$, the following relations hold:

$$\nabla_X \xi = \beta[X - \eta(X)\xi], \tag{25}$$

$$R(X,Y)\xi = \beta^2[\eta(X)Y - \eta(Y)X], \tag{26}$$

$$R(\xi,X)Y = \beta^2[-g(X,Y)\xi + \eta(Y)X], \tag{27}$$

$$S(X,\xi) = -(n-1)\beta^2\eta(X), S(\xi,\xi) = (n-1)\beta^2, \tag{28}$$

$$QX = -(n-1)\beta^2 X, Q\xi = -(n-1)\beta^2\xi. \tag{29}$$

Let us examine the $Q(S,\sigma) = 0$ case for the invariant submanifold M of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{M}$.

Theorem 1. *Let M be the invariant submanifold of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{M}$. If $Q(S,\sigma) = 0$, M is a total geodesic.*

Proof. Let us assume that

$$Q(S,\sigma)(U,V;X,Y) = 0,$$

for all $X, Y, U, V \in \Gamma(TM)$. In this case, we can write

$$\sigma((X \wedge_S Y)U, V) + \sigma(U, (X \wedge_S Y)V) = 0,$$

and so

$$\sigma(S(Y,U)X - S(X,U)Y, V) + \sigma(U, S(Y,V)X - S(X,V)Y) = 0. \tag{30}$$

If we choose $X = V = \xi$ in (30) and use (13), (24), we have

$$(n-1)\beta^2\sigma(U,Y) = 0.$$

Thus, the proof of the theorem is completed. □

Let us examine the $Q(g,\sigma) = 0$ case for the invariant submanifold M of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{M}$.

Theorem 2. *Let M be the invariant submanifold of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{M}$. If $Q(g,\sigma) = 0$, M is a total geodesic.*

Proof. Let us assume that

$$Q\left(g,\sigma\right)\left(U,V;X,Y\right)=0,$$

for all $X,Y,U,V\in\Gamma\left(TM\right).$ In this case, we can write

$$\sigma\left(\left(X\wedge_{g}Y\right)U,V\right)+\sigma\left(U,\left(X\wedge_{g}Y\right)V\right)=0,$$

and so

$$\sigma\left(g\left(Y,U\right)X-g\left(X,U\right)Y,V\right)+\sigma\left(U,g\left(Y,V\right)X-g\left(X,V\right)Y\right)=0. \tag{31}$$

If we choose $Y=U=\xi$ in (31) and use $(1),(2),(24)$, we have

$$\sigma\left(X,V\right)=0.$$

Thus, the proof of the theorem is completed. □

Let us examine the $Q\left(g,\bar{\nabla}\sigma\right)=0$ case for the invariant submanifold M of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{M}$.

Theorem 3. *Let M be the invariant submanifold of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{M}$. If $Q\left(g,\bar{\nabla}\sigma\right)=0$, M is a total geodesic.*

Proof. Let us assume that

$$Q\left(g,\bar{\nabla}\sigma\right)\left(U,V,Z;X,Y\right)=0,$$

for all $X,Y,U,V,Z\in\Gamma\left(TM\right).$ In this case, we can write

$$\left(\bar{\nabla}_{U}\sigma\right)\left(\left(X\wedge_{g}Y\right)V,Z\right)+\left(\bar{\nabla}_{U}\sigma\right)\left(V,\left(X\wedge_{g}Y\right)Z\right)=0,$$

and so

$$\left(\bar{\nabla}_{U}\sigma\right)\left(g\left(Y,V\right)X-g\left(X,V\right)Y,Z\right)+\left(\bar{\nabla}_{U}\sigma\right)\left(V,g\left(Y,Z\right)X-g\left(X,Z\right)Y\right)=0. \tag{32}$$

If we choose $Y=V=\xi$ in (32) and use $(1),(2)$, we have

$$-\left(\bar{\nabla}_{U}\sigma\right)\left(X,Z\right)-\eta\left(X\right)\left(\bar{\nabla}_{U}\sigma\right)\left(\xi,Z\right)+\eta\left(Z\right)\left(\bar{\nabla}_{U}\sigma\right)\left(\xi,X\right)=0. \tag{33}$$

If we use (20) in (33), we get

$$\begin{aligned}&-\nabla_{U}^{\perp}\sigma\left(X,Z\right)+\sigma\left(\nabla_{U}X\right),Z+\sigma\left(X,\nabla_{U}Z\right)\\&+\beta\eta\left(X\right)\sigma\left(U,Z\right)-\beta\eta\left(Z\right)\sigma\left(U,X\right)=0.\end{aligned} \tag{34}$$

If we choose $Z=\xi$ in (34) and use $(1),(24)$, we obtain

$$\sigma\left(U,X\right)=0.$$

Thus, the proof of the theorem is completed. □

Let us examine the $Q\left(S,\bar{\nabla}\sigma\right)=0$ case for the invariant submanifold M of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{M}$.

Theorem 4. *Let M be the invariant submanifold of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{M}$. If $Q\left(S,\bar{\nabla}\sigma\right)=0$, M is a total geodesic.*

Proof. Let us assume that

$$Q\left(S, \bar{\nabla}\sigma\right)(U, V, Z; X, Y) = 0,$$

for all $X, Y, U, V, Z \in \Gamma(TM)$. In this case, we can write

$$\left(\bar{\nabla}_U\sigma\right)((X \wedge_S Y)V, Z) + \left(\bar{\nabla}_U\sigma\right)(V, (X \wedge_S Y)Z) = 0,$$

and so

(35)

$$\left(\bar{\nabla}_U\sigma\right)(S(Y,V)X - S(X,V)Y, Z) + \left(\bar{\nabla}_U\sigma\right)(V, S(Y,Z)X - S(X,Z)Y) = 0.$$

If we choose $Y = Z = \xi$ in (35) and use (28), we have

$$\begin{aligned} &-(n-1)\beta^2\eta(V)\left(\bar{\nabla}_U\sigma\right)(X,\xi) + (n-1)\beta^2\left(\bar{\nabla}_U\sigma\right)(V,X) \\ &+(n-1)\beta^2\eta(X)\left(\bar{\nabla}_U\sigma\right)(V,\xi) = 0. \end{aligned} \tag{36}$$

If we use (20) in (36), we get

$$\begin{aligned} &(n-1)\beta^3\eta(V)\sigma(X,U) + (n-1)\beta^2\left[\nabla_U^{\perp}\sigma(V,X)\right. \\ &\left.-\sigma(\nabla_U V, X) - \sigma(V, \nabla_U X)\right] \\ &+(n-1)\beta^2\eta(X)\left[\nabla_U^{\perp}\sigma(V,\xi) - \sigma(\nabla_U V,\xi) - \sigma(V,\nabla_U\xi)\right]. \end{aligned} \tag{37}$$

If we choose $V = \xi$ in (37) and use (1), (24), (28), we obtain

$$-2(n-1)\beta^3\sigma(U,X) = 0.$$

Thus, the proof of the theorem is completed. □

Let us examine the $Q\left(g, \bar{R}\cdot\sigma\right) = 0$ case for the invariant submanifold M of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{M}$.

Theorem 5. *Let M be the invariant submanifold of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{M}$. If $Q\left(g, \bar{R}\cdot\sigma\right) = 0$, M is a total geodesic.*

Proof. Let us assume that

$$Q\left(g, \bar{R}\cdot\sigma\right)(U, V, Z, W; X, Y) = 0,$$

for all $X, Y, U, V, Z, W \in \Gamma(TM)$. In this case, we can write

$$\begin{aligned} &\left(\bar{R}(X,Y)\cdot\sigma\right)((U \wedge_g V)Z, W) \\ &+\left(\bar{R}(X,Y)\cdot\sigma\right)(Z, (U \wedge_g V)W) = 0, \end{aligned}$$

and

$$\begin{aligned} &\left(\bar{R}(X,Y)\cdot\sigma\right)(g(V,Z)U - g(U,Z)V, W) \\ &+\left(\bar{R}(X,Y)\cdot\sigma\right)(Z, g(V,W)U - g(U,W)V) = 0. \end{aligned} \tag{38}$$

If we choose $Z = U = W = \xi$ in (38) and use (1), (2), we have

$$\left(\bar{R}(X,Y)\cdot\sigma\right)(\eta(V)\xi + V, \xi) = 0. \tag{39}$$

If we use (10) and (24) in (39), we obtain

$$-\beta^2\sigma(V, \eta(X)Y - \eta(Y)X) = 0. \tag{40}$$

If we choose $X=\xi$ in (40) and use $(2),(24)$, we get

$$\sigma(V,Y)=0.$$

Thus, the proof is completed. □

Let us examine the $Q\left(S,\bar{R}\cdot\sigma\right)=0$ case for the invariant submanifold M of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{M}$.

Theorem 6. *Let M be the invariant submanifold of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{M}$. If $Q\left(S,\bar{R}\cdot\sigma\right)=0$, M is a total geodesic.*

Proof. The proof of the theorem can be done in a similar way to the proof of the previous theorem. □

Definition 1. *Let M be an n-dimensional Riemann manifold. The W_1-curvature tensor W_1 is defined by*

$$W_1(X,Y)Z=R(X,Y)Z+\frac{1}{(n-1)}\left[S(Y,Z)X-S(X,Z)Y\right], \tag{41}$$

for all vector fields on M.

Lemma 1. *On an n-dimensional Lorentzian β-Kenmotsu manifold, the W_1-curvature tensor satisfies the following relations:*

$$W_1(\xi,Y)Z=-\beta^2 g(Y,Z)\xi+2\beta^2\eta(Z)Y+\frac{1}{n-1}S(Y,Z)\xi, \tag{42}$$

$$W_1(X,\xi)Z=\beta^2 g(X,Z)\xi-2\beta^2\eta(Z)X-\frac{1}{n-1}S(X,Z)\xi, \tag{43}$$

$$W_1(X,Y)\xi=2\beta^2\left[\eta(X)Y-\eta(Y)X\right], \tag{44}$$

$$\eta\left(W_1(X,Y)Z\right)=2\beta^2 g\left(\eta(Y)X-\eta(X)Y,Z\right). \tag{45}$$

Let us examine the $Q(g,W_1\cdot\sigma)=0$ case for the invariant submanifold M of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{M}$.

Theorem 7. *Let M be the invariant submanifold of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{M}$. If $Q(g,W_1\cdot\sigma)=0$, M is a total geodesic.*

Proof. Let us assume that

$$Q(g,W_1\cdot\sigma)(U,V,Z,W;X,Y)=0,$$

for all $X,Y,U,V,Z,W\in\Gamma(TM)$. In this case, we can write

$$\begin{aligned}&(W_1(X,Y)\cdot\sigma)((U\wedge_g V)Z,W)\\&+(W_1(X,Y)\cdot\sigma)(Z,(U\wedge_g V)W)=0,\end{aligned}$$

and

$$\begin{aligned}&(W_1(X,Y)\cdot\sigma)(g(V,Z)U-g(U,Z)V,W)\\&+(W_1(X,Y)\cdot\sigma)(Z,g(V,W)U-g(U,W)V)=0.\end{aligned} \tag{46}$$

If we choose $Z=U=W=\xi$ in (46) and use (2), we have

$$(W_1(X,Y)\cdot\sigma)(\eta(V)\xi+V,\xi)=0. \tag{47}$$

If we use (24) and (44) in (47), we obtain

$$-2\beta^2\sigma\left(V,\eta\left(X\right)Y-\eta\left(Y\right)X\right)=0. \tag{48}$$

If we choose $X=\xi$ in (48) and use $(2),(24)$, we get

$$\sigma\left(V,Y\right)=0.$$

Thus, the proof is completed. □

Finally, let us examine the $Q\left(S,W_1\cdot\sigma\right)=0$ case for the invariant submanifold M of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{M}$.

Theorem 8. *Let M be the invariant submanifold of the n-dimensional Lorentzian β-Kenmotsu manifold $\tilde{M}$. If $Q\left(S,W_1\cdot\sigma\right)=0$, M is a total geodesic.*

Proof. Let us assume that

$$Q\left(S,W_1\cdot\sigma\right)\left(U,V,Z,W;X,Y\right)=0,$$

for all $X,Y,U,V,Z,W\in\Gamma\left(TM\right)$. In this case, we can write

$$\left(W_1\left(X,Y\right)\cdot\sigma\right)\left(\left(U\wedge_S V\right)Z,W\right)$$
$$+\left(W_1\left(X,Y\right)\cdot\sigma\right)\left(Z,\left(U\wedge_S V\right)W\right)=0,$$

and

$$\begin{aligned}&\left(W_1\left(X,Y\right)\cdot\sigma\right)\left(S\left(V,Z\right)U-S\left(U,Z\right)V,W\right)\\&+\left(W_1\left(X,Y\right)\cdot\sigma\right)\left(Z,S\left(V,W\right)U-S\left(U,W\right)V\right)=0.\end{aligned} \tag{49}$$

If we choose $X=U=W=\xi$ in (49), we have

$$\begin{aligned}&\left(W_1\left(\xi,Y\right)\cdot\sigma\right)\left(S\left(V,Z\right)\xi,\xi\right)-S\left(\xi,Z\right)\left(W_1\left(\xi,Y\right)\cdot\sigma\right)\left(V,\xi\right)\\&+S\left(V,\xi\right)\left(W_1\left(\xi,Y\right)\cdot\sigma\right)\left(Z,\xi\right)-S\left(\xi,\xi\right)\left(W_1\left(\xi,Y\right)\cdot\sigma\right)\left(Z,V\right)=0.\end{aligned} \tag{50}$$

If we choose $Z=\xi$ in (50), we obtain

$$S\left(\xi,\xi\right)\sigma\left(V,W_1\left(\xi,Y\right)\xi\right)=0. \tag{51}$$

If we use (28) and (44) in (51), we have

$$\sigma\left(V,Y\right)=0.$$

Thus, the proof is completed. □

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