UNDER PEER REVIEW

Determining Equations for a Wave Equation Arising Due to Collapse of Shafts in Power Transmission System

Abstract

The study of wave propagation in mechanical systems plays a crucial role in understanding the dynamic behavior of components under stress or failure. One such scenario is the collapse of shafts in power transmission systems, where the sudden failure of structural components can trigger wave-like disturbances that propagate through the system, potentially causing further damage and system-wide disruptions. The ability to predict and analyze these wave phenomena is essential for ensuring the stability, safety, and efficiency of power transmission networks. In this paper, we study a fourth-order nonlinear wave equation which arises due to collapse of shafts in power transmission systems. Lie symmetry analysis is used to derive the corresponding determining equations that describe the symmetries of the system, which can then be used to reduce the problem to lower-order equations, find exact solutions, or gain insights into the underlying dynamics. By applying Lie group analysis, we systematically prolonged the corresponding infinitesimal generator and applied the symmetry condition to the given wave equation to yield the required determining equations.

Keywords

Determining equations, invariance condition, prolongations, infinitesimals

1. Introduction

The Lie symmetry method is a powerful approach for analyzing differential equations, especially when dealing with higher-order equations like the fourth-order wave equation arising in various engineering problems, such as the collapse of shafts in power transmission systems [1]. This method uses group theory to identify symmetries of a differential equation, which in turn allows one to simplify or reduce the equation, obtain invariant solutions, or discover new properties of the system [2]. In general, the Lie symmetry method involves finding continuous transformation groups (Lie groups) that leave a given differential equation invariant. These transformations can be thought of as symmetries of the equation. The essential idea is to determine the infinitesimal generators of the symmetry group and use them to derive the determining equations, which are the necessary conditions that the symmetry generators must satisfy [3].

For an ordinary differential equation (ODE), the determining equations come from the infinitesimal invariance condition. For a partial differential equation (PDE), the symmetry generators can be more complex, involving both space and time derivatives. In both cases, solving the determining equations reveals the symmetries and thus leads to solutions of the original equation or simplifications [4].

A fourth-order wave equation typically takes the following form.

$$\frac{\partial^4 u}{\partial t^4} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

The function $u(x, t)$ represents the displacement or deformation (for instance, of a shaft), and c^2 is a constant related to the wave speed. In engineering contexts such as the collapse of shafts in power transmission systems, this equation models the propagation of stress waves or vibrations through a medium. The symmetry analysis of this equation is crucial for understanding the behavior of the system, including potential simplifications or reductions to lower-dimensional problems. Fourth order wave equation has been researched on by several authors in attempting to find the corresponding determining equations. In addition, the Lie symmetry analysis of fourth-order wave equations has been studied in various contexts, including applied mathematics, physics, and engineering. The focus of these studies has often been on determining the symmetries of such equations and using these symmetries to simplify or find exact solutions to the equations [5]. A number of authors have devoted their work on finding determining equations for related fourth-order wave equations through Lie symmetry analysis as discussed below:

In [6], Olver's foundational work provides a comprehensive framework for applying Lie group theory to both ordinary and partial differential equations. Although this work is general and not focused on fourth-order wave equations specifically, it establishes the essential methodology for symmetry analysis. The book discusses the process of determining symmetry generators, constructing the infinitesimal operator, and solving the corresponding determining equations. This work is foundational in understanding the Lie symmetry method and its application to various differential equations, including wave equations of higher orders. Olver's work is essential because it provides the underlying theory for symmetry analysis, including the methodology for finding determining equations for wave equations of any order, including fourth-order equations.

A detailed exploration of the application of Lie group symmetries to differential equations, particularly focusing on the process of determining symmetries and constructing exact solutions is well executed in the book authored by Bluman, and Kumei [7]. The authors discuss how to find symmetries, formulate the corresponding determining equations, and apply them to specific cases. Although not specifically for power transmission systems, the techniques in this book are directly applicable to any fourth-order wave equations. It provides a general framework for symmetry reduction and solving the associated differential equations. This helps in formulating determining equations for the Lie symmetries of the wave equation governing the collapse of shafts in mechanical systems.

Ibragimov [8], focused on the Lie symmetry analysis of ordinary differential equations and introduced simple methods for the analysis of PDEs of higher orders. The key contribution is its clear explanation of how to construct Lie algebra from the symmetry groups of high-order wave equations. It includes models in mechanics that deal with waves in elastic structures or vibrations; which are analogous to the collapse of shafts in power transmission systems. This provides concrete steps to derive the determining equations and can be directly applied to analyze fourth-order wave equations in mechanical settings. He provides a thorough treatment of the application of Lie groups to both first and higher-order differential equations, with special attention to symmetries and reduction methods. In the context of fourth-order wave equations, this work is crucial because it provides a clear method for identifying the symmetries of such equations and solving the corresponding determining equations. The book includes examples and methodologies that apply directly to wave equations of various orders, including fourth-order wave equations arising in physical systems.

A paper written by Filippo and Katia [9], focused on wave propagation in mechanical systems, specifically analyzing wave propagation in shafts and beams. The authors used Lie symmetry analysis to simplify the equations governing such systems, including fourth-order wave equations that model the vibration and collapse of shafts under stress. This work is significant because it applies Lie symmetry analysis to practical engineering problems. It shows how Lie symmetries can be used to reduce the complexity of fourth-order equations governing wave propagation in power transmission systems, making it relevant to applications in mechanical engineering.

Jervin [10], studied symmetry reductions of higher-order Korteweg-de Vries (KdV) equations, which are related to wave equations in fluid dynamics and nonlinear wave propagation. The paper focused on the Lie symmetry method applied to fourth-order equations similar to the form of the wave equation. The work by Jervin is significant because it demonstrates how symmetry analysis can be applied to fourth-order nonlinear wave equations. The techniques developed in the work are easily transferable to linear fourth-order wave equations, such as those arising in mechanical systems like shafts.

In addition, the work done by Evgenii [11], provided methods for obtaining symmetries of such equations and constructing the corresponding determining equations. The symmetry techniques outlined by Evgenii can be applied to fourth-order wave equations, especially in the context of more complex systems (nonlinear cases). Their approach provided a systematic method for finding determining equations and constructing invariant solutions.

Clarkson and Kruskal [12], explored similarity reductions and the application of symmetry methods to nonlinear wave equations. While the focus was often on first and second-order equations, the methods used can be adapted to fourth-order equations, especially in the study of nonlinear wave propagation. This work is important for understanding how Lie symmetries can be applied to nonlinear wave equations, which are often encountered in the analysis of mechanical systems like shafts under stress. It provides techniques that can be used to reduce the order of fourth-order wave equations and obtain exact solutions.

Grundland, Harnad, Winternitz and Hussain [13-14], focused on applying Lie group methods to reduce the complexity of nonlinear wave equations. He explored the use of symmetry analysis for simplifying both linear and nonlinear wave equations, including fourth-order types. This work is significant in that it applies symmetry reduction methods specifically to nonlinear wave equations, which are often encountered in engineering applications involving shafts, beams, and mechanical systems.

Ibrahim, and Biazar [15], focused specifically on the fourth-order beam equation, a special case of wave equations that describes the bending and vibration of beams and shafts. They use Lie symmetry analysis to identify symmetries and determine exact solutions to the governing equations. This work focused directly on a fourth-order wave equation in the context of structural mechanics. The study applies Lie symmetry analysis to the beam equation, which has similarities to the equations governing shaft behavior in power transmission systems.

Stephen and Gandarias [16], focused on symmetry and reduction techniques for higher-order PDEs in mechanical systems, including the study of wave propagation, vibrations, and elasticity in structures. Their model primarily dealt with structural collapse models, including those in power transmission systems where shaft failure occurs. This offers insights into symmetry reductions and finding determining equations for mechanical systems governed by higher-order wave equations.

An advanced mathematical model for structural mechanics and wave propagation in mechanical systems like power transmission shafts was presented by Craster and Mesarovic [17]. They discussed how wave equations can be derived from elasticity and vibration models. While not focused on Lie symmetry analysis, this text helps establish the physical background for understanding the wave equations governing shaft collapse. The mechanical models discussed provide a framework to apply the symmetry methods found in other references.

Liu and Lou [18], provided a general framework for obtaining symmetries of nonlinear PDEs, including high-order wave equations. The authors discussed how to find the Lie symmetries of systems arising from mechanical dynamics, including shaft vibrations and related nonlinear effects. Their focus was on nonlinear wave equations that are relevant for elastic systems and structural failure (like shaft collapse). This provides advanced techniques for symmetry classification and determining equation generation for higher-order nonlinear PDEs in mechanical applications.

The above references provide both the theoretical framework for Lie symmetry analysis and practical methods for solving the fourth-order wave equations arising from the collapse of shafts in mechanical systems. They provide methods for identifying symmetries, constructing determining equations, and solving complex wave equations that arise in various contexts, including the collapse of shafts in power transmission systems. Key texts such as Olver's foundational work and the contributions by Bluman, Ibragimov, and others lay the groundwork for applying symmetry methods to higher-order wave equations. The work dobe by Filippo and Katia [9]; Jervin [10], show practical applications of these methods in mechanical and engineering contexts. Aminer [20], found the determining equations for a non-linear fourth order ordinary differential equation of the form

$$yy'\left(\left(y\left(y'\right)^{-1}\right)''\right)' = 0$$

which arises in studying the group properties of a linear wave equation in an inhomogeneous medium. However, this equation does not explain the dynamic motion of the collapse of shafts in power transmission systems. In this paper however, we find the determining equations of a special fourth-order wave equation arising due to collapse of shafts in power transmission system modelled by the following equation.

$$y^{(4)} - \frac{4}{3}(y'')^{-1}(y''')^2 = 0$$

2. Preliminaries

The Lie symmetry analysis of the wave equation arising due to the collapse of shafts in power transmission involves using Lie groups of transformations to find the symmetries of the differential equation governing the wave dynamics. These symmetries help in simplifying the problem, reducing the number of independent variables, and finding invariant solutions to the equation [21-23].

The main goal is to find the infinitesimal generators of the Lie group symmetries. These are the differential operators G that act on the dependent and independent variables of the given equation and leave the equation invariant. In order to apply Lie's method to the wave equation, we need to compute the prolonged symmetries. Prolongation involves extending the symmetry operators to act on higher derivatives. The next step is to apply the infinitesimal symmetry operators to the wave equation and require that the equation remains invariant under the transformation. This gives us the determining equations for the components $\xi(x, t)$, $\eta(x, t)$ and $\zeta(x, t)$ [24].

To find the determining equations using Lie symmetry analysis for a fourth-order wave equation, we first need to write down the equation and then identify the infinitesimal generator of the Lie group that leaves the equation invariant. The determining equations are then obtained by applying the invariance condition to the equation, which yields a set of partial differential equations that the coefficients of the infinitesimal generator must satisfy [23]. For instance, given the general fourth-order wave equation of the form:

$$y^{(4)} - c^2 y'' = 0$$

The infinitesimal generator G of a Lie group of transformation would be given by

$$G = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$$

The n^{th} prolongation of the infinitesimal generator G would be given by

$$G^{(n)} = G + \sum_{i=1}^{n} \eta^{(i)} \frac{\partial}{\partial y^{(i)}},$$

Where $\eta^{(i)} = D_x \eta^{(i-1)} - y^{(i)} D_x \xi$ and D_x is the total derivative operator [22-23].

The invariance condition of the given ordinary differential equation is hence given by applying the prolonged infinitesimal generator on the given equation evaluated on its surface and then equated to zero. That is,

$$G^{(4)}\left[y^{(4)} - c^2 y''\right]\Big|_{y^{(4)} - c^2 y''} = 0$$

We then apply the Lie symmetry method to the given ordinary differential equation, computing the prolongations, and using the invariance condition to derive the determining equations for the infinitesimals $\xi(x, t)$ and $\eta(x, t)$ [21].

3. Main Results

Consider a particular form of the fourth order wave equation which arises due to collapse of shafts in power transmission systems; modelled by the equation

$$y^{(4)} - \tfrac{4}{3}(y'')^{-1}(y''')^2 = 0 \qquad (1)$$

We apply the fourth prolongation since our Equation 1 is of 4th order. The *n*th extension according to Mohamed and Leach [21], is given by

$$G^{(n)} = G + \sum_{i=1}^{n}\left\{\beta^{(i)} - \sum_{i=1}^{n}\binom{i}{j} y^{(i+1-j)}\alpha^{(i)}\right\}\frac{\partial}{\partial y^{(i)}} \tag{2}$$

Using Equation 2, we now find the fourth extension, that is, $G^{(4)}$ as follows

$$G^{[4]} = G^{[3]} + \left(\beta^{(4)} - 4y^{(4)}\alpha' - 6y'''\alpha'' - 4y''\alpha''' - y'\alpha^{(4)}\right)\frac{\partial}{\partial y^{(4)}}$$

$$= G^{[2]} + \left(\beta''' - 3y'''\alpha' - 3y''\alpha'' - y'\alpha'''\right)\frac{\partial}{\partial y'''}$$
$$+ \left(\beta^{(4)} - 4y^{(4)}\alpha' - 6y'''\alpha'' - 4y''\alpha''' - y'\alpha^{(4)}\right)\frac{\partial}{\partial y^{(4)}}$$

$$= G^{[1]} + \left(\beta'' - 2y''\alpha' - y'\alpha''\right)\frac{\partial}{\partial y''} + \left(\beta''' - 3y'''\alpha' - 3y''\alpha'' - y'\alpha'''\right)\frac{\partial}{\partial y'''}$$
$$+ \left(\beta^{(4)} - 4y^{(4)}\alpha' - 6y'''\alpha'' - 4y''\alpha''' - y'\alpha^{(4)}\right)\frac{\partial}{\partial y^{(4)}}$$

$$G^{[4]} = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} + (\beta' - \alpha'\,\mathrm{y}')\frac{\partial}{\partial y'} + (\beta'' - 2y''\alpha' - \mathrm{y}'\,\alpha'')\frac{\partial}{\partial y''} \tag{3}$$
$$+ (\beta''' - 3y'''\alpha' - 3y''\alpha'' - \mathrm{y}'\,\alpha''')\frac{\partial}{\partial y'''} + (\beta^{(4)} - 4y^{(4)}\alpha' - 6y'''\alpha'' - 4y''\alpha'''$$
$$- \mathrm{y}'\,\alpha^{(4)})\frac{\partial}{\partial y^{(4)}}$$

Applying the fourth prolongation of the generator on the differential Equation 1, leads to

$$\begin{bmatrix} \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} + (\beta' - \alpha'\,\mathrm{y}')\frac{\partial}{\partial y'} + (\beta'' - 2y''\alpha' - \\ \mathrm{y}'\,\alpha'')\frac{\partial}{\partial y''} + (\beta''' - 3y'''\alpha' - 3y''\alpha'' - \mathrm{y}'\,\alpha''')\frac{\partial}{\partial y'''} \\ + (\beta^{(4)} - 4y^{(4)}\alpha' - 6y'''\alpha'' - 4y''\alpha''' - \mathrm{y}'\,\alpha^{(4)})\frac{\partial}{\partial y^{(4)}} \end{bmatrix}\left[y^{(4)} - \frac{4}{3}(\mathrm{y}'')^{-1}(\mathrm{y}''')^{2}\right] = 0$$

Expanding the above equation results in,

$$\alpha\frac{\partial}{\partial x}\left[y^{(4)}-\frac{4}{3}(y'')^{-1}(y''')^{2}\right]+\beta\frac{\partial}{\partial y}\left[y^{(4)}-\frac{4}{3}(y'')^{-1}(y''')^{2}\right]+(\beta'-\alpha'y')\frac{\partial}{\partial y'}$$
$$\left[y^{(4)}-\frac{4}{3}(y'')^{-1}(y''')^{2}\right]+(\beta''-2y''\alpha'-y'\alpha'')\frac{\partial}{\partial y''}\left[y^{(4)}-\frac{4}{3}(y'')^{-1}(y''')^{2}\right]+$$
$$(\beta'''-3y'''\alpha'-3y''\alpha''-y'\alpha''')\frac{\partial}{\partial y'''}\left[y^{(4)}-\frac{4}{3}(y'')^{-1}(y''')^{2}\right]+(\beta^{(4)}-4y^{(4)}\alpha'$$
$$-6y'''\alpha''-4y''\alpha'''-y'\alpha^{(4)})\frac{\partial}{\partial y^{(4)}}\left[y^{(4)}-\frac{4}{3}(y'')^{-1}(y''')^{2}\right]=0 \tag{4}$$

On differentiating partially Equation 4 we obtain

$$\alpha\left[y^{(5)}+\frac{4}{3}(y'')^{-2}(y''')^{3}-\frac{8}{3}(y''')(y'')^{-1}y^{(4)}\right]+\beta[0]+\left(\beta'-\alpha'y'\right)[0]$$
$$+(\beta''-2y''\alpha'-y'\alpha''')\left[\frac{4}{3}(y'')^{-2}(y''')^{2}-0\right]+\left[\begin{matrix}\beta^{(4)}-4y^{(4)}\alpha'-6y'''\alpha''\\-4y''\alpha'''-y'\alpha^{(4)}\end{matrix}\right][1]=0 \tag{5}$$

But

$$y^{(5)}=\left(y^{(4)}\right)'$$
$$y^{(5)}=\left[\frac{4}{3}(y'')^{-1}(y''')^{2}\right]'$$
$$y^{(5)}=\frac{4}{3}\left[-1(y'')^{-2}(y''')(y''')^{2}+2(y'')^{-1}(y''')^{1}y^{(4)}\right]$$

Therefore,

$$y^{(5)}=-\frac{4}{3}(y'')^{-2}(y''')^{3}+\frac{8}{3}(y'')^{-1}(y''')y^{(4)} \tag{6}$$

Substituting Equation 5 into Equation 6, we obtain the following equations

$$\alpha\left[-\frac{4}{3}(y'')^{-2}(y''')^{3}+\frac{8}{3}(y'')^{-1}(y''')y^{(4)}+\frac{4}{3}(y'')^{-2}(y''')^{3}-\frac{8}{3}(y''')(y'')^{-1}y^{(4)}\right]$$
$$+\beta[0]+\left(\beta'-\alpha'y'\right)[0]+(\beta''-2y''\alpha'-y'\alpha''')\left[\frac{4}{3}(y'')^{-2}(y''')^{2}-0\right]$$
$$+\left[\beta^{(4)}-4y^{(4)}\alpha'-6y'''\alpha''-4y''\alpha'''-y'\alpha^{(4)}\right][1]=0 \tag{7}$$

On simplifying Equation 7 we obtain

$$\frac{4}{3}\left(y''\right)^{-2}\left(y'''\right)^{2}\beta''-\frac{8}{3}\left(y''\right)^{-2}(y''')^{2}(y'')\alpha'-\frac{4}{3}(y')(y'')^{-2}(y''')^{2}\alpha''' \\ +\beta^{(4)}-4y^{(4)}\alpha'-6y'''\alpha''-4y''\alpha'''-y'\alpha^{(4)}=0 \tag{8}$$

The first, second, third, and fourth total derivatives of α and β can therefore be stated in terms of partial derivatives as follows. The primes in Equation 8 correspond to the total derivatives.

$$\alpha'=\frac{\partial\alpha}{\partial x}+y\frac{\partial\alpha}{\partial y}\quad\text{from}\quad d(\alpha)=\left(\frac{\partial\alpha}{\partial x}\right)dx+\left(\frac{\partial\alpha}{\partial y}\right)dy$$

$$\alpha''=\frac{d}{dx}\left(\frac{\partial\alpha}{\partial x}+y'\frac{\partial\alpha}{\partial y}\right)+\frac{d}{dy}\left(\frac{\partial\alpha}{\partial x}+y'\frac{\partial\alpha}{\partial y}\right)y'$$

$$=\frac{\partial^2\alpha}{\partial x^2}+y'\frac{\partial^2\alpha}{\partial y\partial x}+y''\frac{\partial\alpha}{\partial y}+y'\frac{\partial^2\alpha}{\partial x\partial y}+y'^2\frac{\partial^2\alpha}{\partial y^2}+0$$

$$=\frac{\partial^2\alpha}{\partial x^2}+2y'\frac{\partial^2\alpha}{\partial x\partial y}+y'^2\frac{\partial^2\alpha}{\partial y^2}+y''\frac{\partial\alpha}{\partial y} \tag{9}$$

$$\alpha'''=\frac{\partial}{\partial x}\left(\frac{\partial^2\alpha}{\partial x^2}+2y'\frac{\partial^2\alpha}{\partial x\partial y}+y'^2\frac{\partial^2\alpha}{\partial y^2}+y''\frac{\partial\alpha}{\partial y}\right)+y'\frac{\partial}{\partial y}\begin{pmatrix}\frac{\partial^2\alpha}{\partial x^2}+2y'\frac{\partial^2\alpha}{\partial x\partial y}+\\ y''\frac{\partial\alpha}{\partial y}+y'^2\frac{\partial^2\alpha}{\partial y^2}\end{pmatrix}$$

$$=\frac{\partial^3\alpha}{\partial x^3}+\frac{2y'\partial^3\alpha}{2x\partial x\partial y}+\frac{2y''\partial^2\alpha}{\partial x\partial y}+y''\frac{\partial^2\alpha}{\partial y\partial x}+y'''\frac{\partial\alpha}{\partial y}+\frac{y'^2\partial^3\alpha}{\partial x\partial y^2}+2y'\frac{y''\partial^2\alpha}{\partial y^2}$$

$$+\frac{\partial^3\alpha}{\partial x^3}+\frac{2y'\partial^3\alpha}{2x\partial x\partial y}+\frac{2y''\partial^2\alpha}{\partial x\partial y}+y''\frac{\partial^2\alpha}{\partial y\partial x}+y'''\frac{\partial\alpha}{\partial y}+\frac{y'^2\partial^3\alpha}{\partial x\partial y^2}+2y'\frac{y''\partial^2\alpha}{\partial y^2}$$

$$+\frac{y'\partial^3\alpha}{\partial y\partial x^2}+\frac{2y'\partial^3\alpha}{\partial y\partial x\partial y}+0+y'y''\frac{\partial^2\alpha}{\partial y^2}+0+y'^3\frac{\partial^3\alpha}{\partial y^3}+0+\frac{3y'\partial^3\alpha}{\partial x^2\partial y}+\frac{3y''\partial^2\alpha}{\partial x\partial y}$$

$$+y'''\frac{\partial\alpha}{\partial y}+3y'^2\frac{\partial^3\alpha}{\partial x\partial y^2}+3y'y''\frac{\partial^2\alpha}{\partial y^2}+y'^3\frac{\partial^3\alpha}{\partial y^3}.$$

$$\alpha^{(4)}=\frac{\partial}{\partial x}\left(\frac{\partial^3\alpha}{\partial x^3}+\frac{3y'\partial^3\alpha}{\partial x^2\partial y}+3y''\frac{\partial^2\alpha}{\partial x\partial y}+y'''\frac{\partial\alpha}{\partial y}+3y'^2\frac{\partial^3\alpha}{\partial x\partial y^2}+3y'y''\frac{\partial^2\alpha}{\partial y^2}+y'^3\frac{\partial^3\alpha}{\partial y^3}\right)$$

$$+y'\frac{\partial}{\partial y}\left(\frac{\partial^3\alpha}{\partial x^3}+\frac{3y'\partial^3\alpha}{\partial x^2\partial y}+3y''\frac{\partial^2\alpha}{\partial x\partial y}+y'''\frac{\partial\alpha}{\partial y}+3y'^2\frac{\partial^3\alpha}{\partial x\partial y^2}+3y'y''\frac{\partial^2\alpha}{\partial y^2}+y'^3\frac{\partial^3\alpha}{\partial y^3}\right)$$

$$=\frac{\partial^4\alpha}{\partial x^4}+3y'\frac{\partial^4\alpha}{\partial x^3\partial y}+3y''\frac{\partial^3\alpha}{\partial x^2\partial y}+3y''\frac{\partial^3\alpha}{\partial x^2\partial y}+3y'''\frac{\partial^2\alpha}{\partial x\partial y}+y'''\frac{\partial^2\alpha}{\partial x\partial y}+y^{(4)}\frac{\partial\alpha}{\partial y}$$

$$+3y'^2\frac{\partial^4\alpha}{\partial x^2\partial y^2}+6y'y''\frac{\partial^3\alpha}{\partial x\partial y^2}+3y'y''\frac{\partial^3\alpha}{\partial x\partial y^2}+3y'y'''\frac{\partial^2\alpha}{\partial y^2}+3y''^2\frac{\partial^2\alpha}{\partial y^2}+y'^3\frac{\partial^4\alpha}{\partial x\partial y^3}$$

$$+3y'^2\frac{\partial^3\alpha}{\partial y^3}+y'\frac{\partial^4\alpha}{\partial y\partial x^3}+3y'^2\frac{\partial^4\alpha}{\partial x^2\partial y^2}+3y'y''\frac{\partial^3\alpha}{\partial x\partial y^2}+y'y'''\frac{\partial^2\alpha}{\partial y^2}+3y'^3\frac{\partial^4\alpha}{\partial x\partial y^3}$$

$$+3y'^2y''\frac{\partial^3\alpha}{\partial y^3}+y'^4\frac{\partial^4\alpha}{\partial y^4}.$$

$$=\frac{\partial^4\alpha}{\partial x^4}+4y'\frac{\partial^4\alpha}{\partial x^3\partial y}+6y''\frac{\partial^3\alpha}{\partial x^2\partial y}+4y'''\frac{\partial^2\alpha}{\partial x\partial y}+y^{(4)}\frac{\partial\alpha}{\partial y}+3y'\frac{\partial^4\alpha}{\partial x^2\partial y^2}$$

$$+9y'y''\frac{\partial^3\alpha}{\partial x\partial y^2}+4y'y'''\frac{\partial^2\alpha}{\partial y^2}+3y''^2\frac{\partial^2\alpha}{\partial y^2}+4y'\frac{\partial^4\alpha}{\partial x\partial y^3}+6y'^2y''\frac{\partial^3\alpha}{\partial y^3}$$

$$+3y'^2\frac{\partial^4\alpha}{\partial x^2\partial y^2}+3y'y''\frac{\partial^3\alpha}{\partial x\partial y^2}+y'^4\frac{\partial^4\alpha}{\partial y^4}$$

And

$$\beta'=\frac{\partial\beta}{\partial x}+y'\frac{\partial\beta}{\partial y}\quad\text{from}\quad d(\beta)=\left(\frac{\partial\beta}{\partial x}\right)dx+\left(\frac{\partial\beta}{\partial y}\right)dy$$

$$\beta''=\frac{d}{dx}\left(\frac{\partial\beta}{\partial x}+y'\frac{\partial\beta}{\partial y}\right)+\frac{d}{dy}\left(\frac{\partial\beta}{\partial x}+y'\frac{\partial\beta}{\partial y}\right)y'$$

$$=\frac{\partial^2\beta}{\partial x^2}+y'\frac{\partial^2\beta}{\partial y\partial x}+y''\frac{\partial\beta}{\partial y}+y'\frac{\partial^2\beta}{\partial x\partial y}+y'^2\frac{\partial^2\beta}{\partial y^2}+0$$

$$=\frac{\partial^2\beta}{\partial x^2}+2y'\frac{\partial^2\beta}{\partial x\partial y}+y'^2\frac{\partial^2\beta}{\partial y^2}+y''\frac{\partial\beta}{\partial y}\tag{10}$$

$$\beta''' = \frac{\partial}{\partial x}\left(\frac{\partial^2\beta}{\partial x^2} + 2y'\frac{\partial^2\beta}{\partial x\partial y} + y'^2\frac{\partial^2\beta}{\partial y^2} + y''\frac{\partial\beta}{\partial y}\right) + y'\frac{\partial}{\partial y}\begin{pmatrix}\frac{\partial^2\beta}{\partial x^2} + 2y'\frac{\partial^2\beta}{\partial x\partial y} + \\ y''\frac{\partial\beta}{\partial y} + y'^2\frac{\partial^2\beta}{\partial y^2}\end{pmatrix}$$

$$= \frac{\partial^3\beta}{\partial x^3} + 2y'\frac{\partial^3\beta}{\partial x\partial x\partial y} + 2y''\frac{\partial^2\beta}{\partial x\partial y} + y''\frac{\partial^2\beta}{\partial y\partial x} + y'''\frac{\partial\beta}{\partial y} + y'^2\frac{\partial^3\beta}{\partial x\partial y^2} + 2y'y''\frac{\partial^2\beta}{\partial y^2}$$

$$+\frac{\partial^3\beta}{\partial x^3} + \frac{2y'\partial^3\beta}{\partial x\partial x\partial y} + \frac{2y''\partial^2\beta}{\partial x\partial y} + y''\frac{\partial^2\beta}{\partial y\partial x} + y'''\frac{\partial\beta}{\partial y} + \frac{y'^2\partial^3\beta}{\partial x\partial y^2} + 2y'\frac{y''\partial^2\beta}{\partial y^2}$$
$$+\frac{y'\partial^3\beta}{\partial y\partial x^2} + \frac{2y'\partial^3\beta}{\partial y\partial x\partial y} + 0 + y'y''\frac{\partial^2\beta}{\partial y^2} + 0 + y'^3\frac{\partial^3\beta}{\partial y^3} + 0 + \frac{3y'\partial^3\beta}{\partial x^2\partial y} + \frac{3y''\partial^2\beta}{\partial x\partial y}$$
$$+y'''\frac{\partial\beta}{\partial y} + 3y'^2\frac{\partial^3\beta}{\partial x\partial y^2} + 3y'y''\frac{\partial^2\beta}{\partial y^2} + y'^3\frac{\partial^3\beta}{\partial y^3}.$$

$$\beta^{(4)} = \frac{\partial}{\partial x}\left(\frac{\partial^3\beta}{\partial x^3} + \frac{3y'\partial^3\beta}{\partial x^2\partial y} + 3y''\frac{\partial^2\beta}{\partial x\partial y} + y'''\frac{\partial\beta}{\partial y} + 3y'^2\frac{\partial^3\beta}{\partial x\partial y^2} + 3y'y''\frac{\partial^2\beta}{\partial y^2} + y'^3\frac{\partial^3\beta}{\partial y^3}\right)$$
$$+y'\frac{\partial}{\partial y}\left(\frac{\partial^3\beta}{\partial x^3} + \frac{3y'\partial^3\beta}{\partial x^2\partial y} + 3y''\frac{\partial^2\beta}{\partial x\partial y} + y'''\frac{\partial\beta}{\partial y} + 3y'^2\frac{\partial^3\beta}{\partial x\partial y^2} + 3y'y''\frac{\partial^2\beta}{\partial y^2} + y'^3\frac{\partial^3\beta}{\partial y^3}\right)$$

$$= \frac{\partial^4\beta}{\partial x^4} + 3y'\frac{\partial^4\beta}{\partial x^3\partial y} + 3y''\frac{\partial^3\beta}{\partial x^2\partial y} + 3y''\frac{\partial^3\beta}{\partial x^2\partial y} + 3y'''\frac{\partial^2\beta}{\partial x\partial y} + y'''\frac{\partial^2\beta}{\partial x\partial y} + y^{(4)}\frac{\partial\beta}{\partial y}$$

$$+3y'^2\frac{\partial^4\beta}{\partial x^2\partial y^2} + 6y'y''\frac{\partial^3\beta}{\partial x\partial y^2} + 3y'y''\frac{\partial^3\beta}{\partial x\partial y^2} + 3y'y'''\frac{\partial^2\beta}{\partial y^2} + 3y''^2\frac{\partial^2\beta}{\partial y^2} + y'^3\frac{\partial^4\beta}{\partial x\partial y^3}$$
$$+3y'^2\frac{\partial^3\beta}{\partial y^3} + y'\frac{\partial^4\beta}{\partial y\partial x^3} + 3y'^2\frac{\partial^4\beta}{\partial x^2\partial y^2} + 3y'y''\frac{\partial^3\beta}{\partial x\partial y^2} + y'y'''\frac{\partial^2\beta}{\partial y^2} + 3y'^3\frac{\partial^4\beta}{\partial x\partial y^3}$$
$$+3y'^2y''\frac{\partial^3\beta}{\partial y^3} + y'^4\frac{\partial^4\beta}{\partial y^4}.$$

$$=\frac{\partial^4\beta}{\partial x^4}+4y'\frac{\partial^4\beta}{\partial x^3\partial y}+6y''\frac{\partial^3\beta}{\partial x^2\partial y}+4y'''\frac{\partial^2\beta}{\partial x\partial y}+y^{(4)}\frac{\partial\beta}{\partial y}+3y'\frac{\partial^4\beta}{\partial x^2\partial y^2}$$
$$+9y'y''\frac{\partial^3\beta}{\partial x\partial y^2}+4y'y'''\frac{\partial^2\beta}{\partial y^2}+3y''^2\frac{\partial^2\beta}{\partial y^2}+4y'\frac{\partial^4\beta}{\partial x\partial y^3}+6y'^2y''\frac{\partial^3\beta}{\partial y^3}$$
$$+3y'^2\frac{\partial^4\beta}{\partial x^2\partial y^2}+3y'y''\frac{\partial^3\beta}{\partial x\partial y^2}+y'^4\frac{\partial^4\beta}{\partial y^4}$$

Substituting the above results into Equation 10, we obtain the following

$$\frac{4}{3}(y'')^{-2}(y''')^2\left(\frac{\partial^2\beta}{\partial x^2}+2y'\frac{\partial^2\beta}{\partial x\partial y}+y'^2\frac{\partial^2\beta}{\partial y^2}+y''\frac{\partial\beta}{\partial y}\right)-\frac{8}{3}(y'')^{-1}(y''')^2$$
$$+\left(\frac{\partial\alpha}{\partial x}+y'\frac{\partial\alpha}{\partial y}\right)-\frac{4}{3}y'(y'')^{-2}(y''')^2\left(\frac{\partial^2\alpha}{\partial x^2}+2y'\frac{\partial^2\alpha}{\partial x\partial y}+y'^2\frac{\partial^2\alpha}{\partial y^2}+y''\frac{\partial\alpha}{\partial y}\right)$$
$$+\frac{\partial^4\beta}{\partial x^4}+4y'\frac{\partial^4\beta}{\partial x^3\partial y}+6y''\frac{\partial^3\beta}{\partial x^2\partial y}+4y'''\frac{\partial^2\beta}{\partial x\partial y}+y^{(4)}\frac{\partial\beta}{\partial y}+3y'^2\frac{\partial^4\beta}{\partial x^2\partial y^2}$$
$$+9y'y''\frac{\partial^3\beta}{\partial x\partial y^2}+6y''\frac{\partial^3\beta}{\partial x^2\partial y}+4y'y'''\frac{\partial^2\beta}{\partial y^2}+3y''^2\frac{\partial^2\beta}{\partial y^2}+4y'^3\frac{\partial^4\beta}{\partial x\partial y^3}$$
$$+6y'^2y''\frac{\partial^3\beta}{\partial y^3}+3y'^2\frac{\partial^4\beta}{\partial x^2\partial y^2}+3y'y''\frac{\partial^3\beta}{\partial x\partial y^2}+y'^4\frac{\partial^4\beta}{\partial y^4}$$

$$-4y^{(4)}\left(\frac{\partial\alpha}{\partial x}+y'\frac{\partial\alpha}{\partial y}\right)-6y''\left(\frac{\partial^3\alpha}{\partial x^3}+3y'\frac{\partial^3\alpha}{\partial x^2\partial y}\right)+3y''\frac{\partial^3\alpha}{\partial x\partial y^2}+y'''\frac{\partial\alpha}{\partial y}+3y'^2\frac{\partial^3\alpha}{\partial x\partial y^2}$$
$$+3y'y''\frac{\partial^3\alpha}{\partial y^3}+y'^3\frac{\partial^3\alpha}{\partial y^3}-4y''\left(\frac{\partial^3\alpha}{\partial x^3}+3y'\frac{\partial^3\alpha}{\partial x^2\partial y}\right)+3y''\frac{\partial^2\alpha}{\partial x\partial y}+y'''\frac{\partial\alpha}{\partial y}+3y'^3\frac{\partial^3\alpha}{\partial x\partial y^2}$$
$$+3y'y''\frac{\partial^2\alpha}{\partial y^2}+3y'^3\frac{\partial^3\alpha}{\partial y^3}-y'\left(\frac{\partial^4\alpha}{\partial x^4}+4y'\frac{\partial^4\alpha}{\partial x^3\partial y}\right)+6y''\frac{\partial^3\alpha}{\partial x^2\partial y}+4y'''\frac{\partial^2\alpha}{\partial x\partial y}+y^{(4)}\frac{\partial\alpha}{\partial y}$$
$$+3y'^2\frac{\partial^4\alpha}{\partial x^2\partial y^2}+9y'y''\frac{\partial^3\alpha}{\partial x\partial y^2}+4y'y''\left(\frac{\partial^2\alpha}{\partial y^2}+3y''^2\frac{\partial^2\alpha}{\partial y^2}\right)+4y'^3\frac{\partial^4\alpha}{\partial x\partial y^3}+6y'^2y''\frac{\partial^3\alpha}{\partial y^3}$$

$$+3y'^2\frac{\partial^4\alpha}{\partial x^2\partial y^2}+3y'y''\frac{\partial^3\alpha}{\partial x\partial y^2}+y'^4\frac{\partial^4\alpha}{\partial y^4} \qquad (11)$$

Expanding Equation 11 and simplifying results in

$$\frac{4}{3}(y'')^{-2}(y''')^{2}\frac{\partial^{2}\beta}{\partial x^{2}}+\frac{4}{3}(y'')^{-2}(y''')^{2}\left(2y'\frac{\partial^{2}\beta}{\partial x\partial y}\right)+\frac{4}{3}(y'')^{-2}(y''')^{2}\left(y'^{2}\right)\frac{\partial^{2}\beta}{\partial y^{2}}$$
$$+\frac{4}{3}(y'')^{-2}(y''')^{2}\left(y''\right)\frac{\partial\beta}{\partial y}-\frac{8}{3}(y'')^{-1}(y''')^{2}\frac{\partial\alpha}{\partial x}-\frac{8}{3}(y'')^{-1}(y''')^{2}y'\frac{\partial\alpha}{\partial y}$$
$$-\frac{4}{3}y'(y'')^{-2}(y''')^{2}\frac{\partial^{2}\alpha}{\partial x^{2}}-\frac{4}{3}y'(y'')^{-2}(y''')^{2}(2y')\frac{\partial^{2}\alpha}{\partial x\partial y}-\frac{4}{3}y'(y'')^{-2}(y''')^{2}y'^{2}\frac{\partial^{2}\alpha}{\partial y^{2}}$$
$$-\frac{4}{3}y'(y'')^{-2}(y''')^{2}y''\frac{\partial\alpha}{\partial y}$$

$$+\frac{\partial^{4}\beta}{\partial x^{4}}+4y'\frac{\partial^{4}\beta}{\partial x^{3}\partial y}+6y''\frac{\partial^{3}\beta}{\partial x^{2}\partial y}+4y'''\frac{\partial^{2}\beta}{\partial x\partial y}+y^{(4)}\frac{\partial\beta}{\partial y}+3y'^{2}\frac{\partial^{4}\beta}{\partial x^{2}\partial y^{2}}+9y'y''\frac{\partial^{3}\beta}{\partial x\partial y^{2}}$$
$$+6y''\frac{\partial^{3}\beta}{\partial x^{2}\partial y}+4y'y'''\frac{\partial^{2}\beta}{\partial y^{2}}+3y''^{2}\frac{\partial^{2}\beta}{\partial y^{2}}+4y'^{3}\frac{\partial^{4}\beta}{\partial x\partial y^{3}}+6y'^{2}y''\frac{\partial^{3}\beta}{\partial y^{3}}+3y'^{2}\frac{\partial^{4}\beta}{\partial x^{2}\partial y^{2}}$$
$$+3y'y''\frac{\partial^{3}\beta}{\partial x\partial y^{2}}+y'^{4}\frac{\partial^{4}\beta}{\partial y^{4}}-4y^{(4)}\frac{\partial\alpha}{\partial x}-4y^{(4)}y'\frac{\partial\alpha}{\partial y}-6y''\frac{\partial^{3}\alpha}{\partial x^{3}}-18y''y'^{2}\frac{\partial^{3}\alpha}{\partial x^{2}\partial y}$$

$$-18(y'')^{2}\frac{\partial^{2}\alpha}{\partial x\partial y}-6(y'')(y''')\frac{\partial\alpha}{\partial y}-18y''y'^{2}\frac{\partial^{3}\alpha}{\partial x\partial y^{2}}-18(y'')^{2}y'\frac{\partial^{2}\alpha}{\partial y^{2}}-6y''(y')^{3}\frac{\partial^{3}\alpha}{\partial y^{3}}$$

$$-4(y'')\frac{\partial^{3}\alpha}{\partial x^{3}}-12(y'')y'\frac{\partial^{3}\alpha}{\partial x^{3}\partial y}-12(y'')^{2}\frac{\partial^{2}\alpha}{\partial x\partial y}-4(y'')(y''')\frac{\partial\alpha}{\partial y}-12y'^{2}y''\frac{\partial^{3}\alpha}{\partial x\partial y^{2}}$$

$$-12y'(y'')^{2}\frac{\partial^{2}\alpha}{\partial y^{2}}-4(y')^{3}y''\frac{\partial^{3}\alpha}{\partial y^{3}}-y'\frac{\partial^{4}\alpha}{\partial x^{4}}-4(y')^{2}\frac{\partial^{4}\alpha}{\partial x^{3}\partial y}-6y'y''\frac{\partial^{3}\alpha}{\partial x^{3}\partial y}-4y'y'''\frac{\partial^{2}\alpha}{\partial x\partial y}$$

$$-y'y^{(4)}\frac{\partial\alpha}{\partial y}-3(y')^{3}\frac{\partial^{4}\alpha}{\partial x^{2}\partial y^{2}}-9(y')^{2}y''\frac{\partial^{3}\alpha}{\partial x\partial y^{2}}-4(y')^{2}y''\frac{\partial^{2}\alpha}{\partial y^{2}}-3y'(y'')^{2}\frac{\partial^{2}\alpha}{\partial x^{2}}-4(y')^{4}$$

$$\frac{\partial^{4}\alpha}{\partial x\partial y^{3}}-6(y')^{3}y''\frac{\partial^{3}\alpha}{\partial y^{3}}-3(y')^{3}\frac{\partial^{4}\alpha}{\partial x^{2}\partial y^{2}}-3(y')^{2}y''\frac{\partial^{3}\alpha}{\partial x\partial y^{2}}-(y')^{5}\frac{\partial^{4}\alpha}{\partial y^{4}}=0 \quad (12)$$

In x, y, y', y'' and y''', Equation 12 is an identity, meaning that it holds true for any arbitrary chance of x, y, y', y'' and y'''. Being functions of only x and y, α and β must equal zero for the coefficient of the powers of y', y'', y''', and their combinations. The following partial differential equations, what we refer to as determining equations, are obtained

$$(y')^3(y'')^{-2}(y''')^3 : \frac{4}{3}\frac{\partial^2\alpha}{\partial y^2} = 0 \tag{13}$$

$$(y')^2(y'')^{-2}(y''')^3 : \frac{4}{3}\frac{\partial^2\beta}{\partial y^2} - \frac{8}{3}\frac{\partial^2\alpha}{\partial x\partial y} = 0 \tag{14}$$

$$(y')^1(y'')^{-2}(y''')^3 : -\frac{4}{3}\frac{\partial^2\alpha}{\partial x^2} + \frac{8}{3}\frac{\partial^2\beta}{\partial x\partial y} = 0 \tag{15}$$

$$(y')^0(y'')^{-2}(y''')^3 : \frac{\partial^2\beta}{\partial x^2} \tag{16}$$

Equation 13, Equation 14, Equation 15, and Equation 16, are the required determining equations for our fourth order wave equation

$$\boldsymbol{y^{(4)} - \frac{4}{3}(y'')^{-1}(y''')^2 = 0}$$

4 Conclusions

The Lie symmetry technique offers a systematic way to tackle complex fourth-order wave equations, such as one arising due to the collapse of shafts in power transmission. By applying this method, we have obtained the determining equations that describe the symmetries of the system, which can then be used to reduce the problem to lower-order equations, find exact solutions, or gain insights into the underlying dynamics. Several key works have demonstrated the utility of Lie symmetry analysis in reducing the complexity of such higher-order wave equations, both in the theoretical and applied contexts. This analysis is crucial in understanding the behavior of the system under various transformations and initial conditions.

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