Normalized solutions for a quasilinear Schrödinger Choquard equation with exponential critical growth in $\mathbb{R}^2$

Abstracts: In this paper, we are concerned with normalized solutions to the following quasilinear Schrödinger Choquard equation

$$-\Delta u - u\Delta u^2 + \lambda u = (I_\alpha * F(u))\, f(u), \quad \text{in } \mathbb{R}^2,$$

with prescribed mass

$$\int_{\mathbb{R}^2} |u|^2 \, dx = a^2,$$

where $a > 0$, $\lambda \in \mathbb{R}, \alpha \in (0,2), I_\alpha$ denotes the Riesz potential, $*$ denotes the convolution opertor, and the nonlinearity f has an exponential critical growth in the sense of Trudinger-Moser inequality. Using Perturbation method and variational methods with Pohozaev manifold, we can avoid the nondifferentiability of the quasilinear term $u\Delta u^2$ and prove the existence of normalized solutions with some further assumption.

Keywords: Normalized solutions; Quasilinear Schrödinger equation; Choquard equation; Exponential critical growth.

1 Introduction

The following generic quasilinear problems have described several physical situations of the form

$$\begin{cases} i\partial_t \psi = -\Delta\psi - \psi l'(|\psi|^2)\Delta l(|\psi|^2) + f(|\psi|^2)\psi, & \text{in } \mathbb{R}^+ \times \mathbb{R}^2, \\ \psi(0,x) = \psi_0(x), & \text{in } \mathbb{R}^2, \end{cases} \tag{1.1}$$

where l, f are given functions, i denotes the imaginary unit, and $\psi : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{C}$ is a complex real function. It is well known that when $l(s) = \sqrt{s}$, problem (1.1) appears in plasma physics and fluid mechanics [17, 27], also in the theory of Heisenberg ferromagnet and in condensed matter theory [23], and the dynamic properties are closely linked to l and f. While, in this article, we focus on the particular case $l(s) = s$, that is

$$\begin{cases} i\partial_t \psi = -\Delta\psi - \psi\Delta(|\psi|^2) + f(|\psi|^2)\psi, & \text{in } \mathbb{R}^+ \times \mathbb{R}^2, \\ \psi(0,x) = \psi_0(x), & \text{in } \mathbb{R}^2. \end{cases} \tag{1.2}$$

A stationary wave solution is a solution of the form $\psi(t,x) = e^{-i\lambda t}u(x)$, where $\lambda \in \mathbb{R}$ is a parameter and $u(x) : \mathbb{R}^2 \to \mathbb{R}$ is a time-independent function to be founded. Substitute $\psi(t,x) = e^{-i\lambda t}u(x)$ into (1.2), we obtain the following stationary equation

$$-\Delta u - u\Delta u^2 = \lambda u + f(u), \quad \text{in} \quad \mathbb{R}^2. \tag{1.3}$$

For some fixed values of λ, a nontrivial solution of (1.3) is obtained as a critical point of the functional $J_\lambda : H^{1,2}(\mathbb{R}^2) \to \mathbb{R}$, which is given by

$$J_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\, dx + \int_{\mathbb{R}^2} |u|^2|\nabla u|^2\, dx - \frac{1}{2}\int_{\mathbb{R}^2} \lambda |u|^2 dx - \int_{\mathbb{R}^2} F(u)\, dx,$$

where $F(t) = \displaystyle\int_0^t f(s)\, ds$, the primitive function of $f(t)$, on the natural space

$$H = \left\{ u \in H^{1,2}(\mathbb{R}^2) : \int_{\mathbb{R}^2} |u|^2|\nabla u|^2\, dx < +\infty \right\},$$

another important way to find the nontrivial solutions for (1.3) is to search for solutions with prescribed mass, that is

$$\begin{cases} -\Delta u - u\Delta u^2 = \lambda u + f(u), \quad \text{in} \quad \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} |u|^2\, dx = a^2, \end{cases} \tag{1.4}$$

in this case, $\lambda \in \mathbb{R}$ is part of the unknown. Moreover, it can be obtained by looking for critical points of the corresponding energy functional:

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\, dx + \int_{\mathbb{R}^2} |u|^2|\nabla u|^2\, dx - \int_{\mathbb{R}^2} F(u)\, dx \tag{1.5}$$

on the L^2-sphere

$$\widetilde{S}(a) = \left\{ u \in H : \int_{\mathbb{R}^2} |u|^2\, dx = a^2 \right\},$$

which has particular difficulties. To derive the Palais-Smale sequence, one needs new variational methods, because the derived Palais-Smale sequence may not be bounded; even if the Palais-Smale sequence is bounded, the weak limit may not be contained in the L^2-sphere (even in the radical case). Such difficulties make the study of normalized solutions for (1.4) much more complicated than (1.3) with prescribed $\lambda \in \mathbb{R}$.

One quasilinear term

$$V(u) := \int_{\mathbb{R}^2} |u|^2|\nabla u|^2\, dx$$

in (1.5) has put forward a new problem: it is not differentiable in the space H. To overcome this difficulty, during the last ten years, various arguments have been put forward on standing wave solutions, while very few results are known about equations of the normalized solutions. Using the minimization methods [26], Nehari manifold approach [18], change variables [9,20] methods, and perturbation method in a series of paper [19, 21, 22], that recovers the differentiability by considering a perturbed functional on a smaller function space, one can obtain the standing wave soulutions but not normalized solution. To the best of our knowledge, Houwang Li and Wenming Zou [16] discussed the normalized solutions for quasilinear problem (1.4) with $f(u) = |u|^{p-2}u$ satisfies $p > 4 + \frac{4}{N}$, and $a > 0$.

Motivated by the results above, considering that there are few results on normalized solutions for the quasilinear Schrödinger equation with exponential critical growth, in this paper, we focus on the system (1.4) discussed before, where $a \in (0, 1)$, $\lambda \in \mathbb{R}$ and f has an exponential critical growth. We recall that in $\mathbb{R}^2$, the natural growth restriction on function f is given by the inequality of Trudinger and Moser [24, 34].

In this paper, we assume that f is a continuous function that satisfies the following conditions:

(f_1) $\lim\limits_{t\to 0} \dfrac{|f(t)|}{|t|^\tau} = 0$, for some $\tau > 2 + \dfrac{\alpha}{2}$;

(f_2) f has γ_0- exponential critical growth, *i.e.*, there exists $\gamma_0 > 0$ such that

$$\lim_{|t|\to+\infty} \frac{|f(t)|}{e^{\gamma t^2}} = \begin{cases} 0, & \text{for } \gamma > \gamma_0, \\ +\infty, & \text{for } 0 < \gamma < \gamma_0, \end{cases}$$

(f_3) There exists a constant $\kappa > 3 + \dfrac{\alpha}{2}$ such that

$$0 < \kappa F(t) \le t f(t), \text{ for all } t \in \mathbb{R}\backslash\{0\}, \text{ where } F(t) = \int_0^t f(s)ds;$$

(f_4) There exist constants $\sigma > 3 + \dfrac{\alpha}{2}$ and $\mu > 0$ such that

$$F(t) \ge \mu |t|^\sigma, \text{ for all } t \in \mathbb{R}\backslash\{0\},$$

Our main result is as follows:

Theorem 1.1. *Assume that f satisfies $(f_1) - (f_4)$. If $a^2 < \frac{(2+\alpha)\pi}{\gamma_0}$, then there exists $\mu^* = \mu^*(a) > 0$ such that problem (1.4) admits a couple of normalized soluition $(u_a, \lambda_a) \in H^{1,2}(\mathbb{R}^2) \times \mathbb{R}$ of weak solution, u_a is a radially symmetric function, and $\lambda_a < 0$ for all $\mu \ge \mu^*$.*

Remark 1.1. A typical example satisfying $(f_1) - (f_4)$ is

$$f(t) = \mu |t|^{p-2} t e^{\alpha_0} |t|^2, \text{ for } p > 3 + \frac{\alpha}{2}, \text{ and all } t \in \mathbb{R}.$$

The organization of this paper is as follows: in Section 2, we state some preliminary lemmas and perturbation settings. In Section 3, we use the mountain-pass arguments to construct a bounded (PS) sequence. In Section 4, we prove the existence of critical points for perturbation functional. In Section **??**, we study the convergence of critical points for the perturbation functional as $\eta \to 0^+$. Section 5 is devoted to the proof of Theorem 1.1.

Notations:

- Write $|u|_p := \left(\int_{\mathbb{R}^2} |u|^p dx \right)^{\frac{1}{p}}$ with $1 \le p < \infty$.
- Denote $H^{1,2}(\mathbb{R}^2) := \left\{ u \in L^2(\mathbb{R}^2) \mid Du \in L^2(\mathbb{R}^2) \right\}$ as the Sobolev space with the norm

$$\|u\| := \left(\int_{\mathbb{R}^2} |\nabla u|^2 dx + \int_{\mathbb{R}^2} |u|^2 dx \right)^{\frac{1}{2}}.$$

• Use " $\to$ " and " $\rightharpoonup$ " respectively to denotes the strong and weak convergence in the related function space.
• Define $B_R^c := \{x \in \mathbb{R}^2 : |x| > R\}$.
• Use " $C_1, C_2, C_3 \cdots$ " denotes any positive constants (possibly different).
• Denote $o_n(1)$ a real sequence with $o_n(1) \to 0$ as $n \to +\infty$.

Next, we will introduce the following Gagliardo-Sobolev inequalitity [1]: for any $p > 2$,

$$|u|_p \le C_p |\nabla u|_2^{\gamma_p} |u|_2^{1-\gamma_p}, \quad \forall u \in H^{1,2}(\mathbb{R}^2), \tag{1.6}$$

where

$$\gamma_p := 2\left(\frac{1}{2} - \frac{1}{p}\right).$$

and the following Gagliardo-Nirenberg-type inequality [16]:

$$\int_{\mathbb{R}^2} |u|^p \, dx \le C_p \left(\int_{\mathbb{R}^2} |u|^2 \, dx \right) \left(4 \int_{\mathbb{R}^2} |u|^2 |\nabla u|^2 \, dx \right)^{\frac{p-2}{4}}, \quad \forall u \in H^{1,2}(\mathbb{R}^2). \tag{1.7}$$

2 Preliminaries

2.1 Preliminary lemmas

We start our study recalling that by $(f_1) - (f_3)$ and $f(t)$ has critical exponential growth at $+\infty$ with critical exponential γ_0, then fix $q > 1 + \frac{\alpha}{2}$, $\tau > 2 + \frac{\alpha}{2}$, for any $\varepsilon > 0$ and $\gamma > \gamma_0$ close to γ_0, there exists a constant $C > 0$ which depends on ε and μ such that

$$|f(t)| \le \varepsilon |t|^{\tau} + C_{\varepsilon,\mu} |t|^{q-1} (e^{\gamma t^2} - 1) \quad \text{for all } t \in \mathbb{R}, \tag{2.1}$$

and, it is easy to see that

$$|F(t)| \le \varepsilon |t|^{\tau+1} + C_{\varepsilon,\mu} |t|^{q} (e^{\gamma t^2} - 1) \quad \text{for all } t \in \mathbb{R}. \tag{2.2}$$

Now, we recall the following version of Trudinger-Moser inequality as stated in [7].

Lemma 2.1. *(i) If $\gamma > 0$ and $u \in H^{1,2}(\mathbb{R}^2)$, then*

$$\int_{\mathbb{R}^2} (e^{\gamma u^2} - 1) dx < +\infty.$$

(ii) Moreover, if $|\nabla u|_2^2 \le 1$, $|u|_2^2 \le M < +\infty$, and $0 < \gamma < 4\pi$, then, there exists a constant $C > 0$ which depends only on M and γ, such that

$$\int_{\mathbb{R}^2} (e^{\gamma u^2} - 1) dx \le C_{\gamma,M}.$$

And the Hardy-Littlewood-Sobolev inequality, see

Lemma 2.2. *Let $t, r > 1, 0 < \alpha < 2$, with $\frac{1}{t} + \frac{2-\alpha}{2} + \frac{1}{r} = 2, f \in L^t(\mathbb{R}^2)$ and $g \in L^r(\mathbb{R}^2)$. Then, there exists a sharp constant C which depends on t, α, r, such that*

$$\int_{\mathbb{R}^2} (I_\alpha * f) g dx \leq C_{t,\alpha,r} |f|_t |g|_r.$$

According to Lemma 2.2, we arrive at

$$\int_{\mathbb{R}^2} (I_\alpha * F(u)) F(u) dx$$

is well-defined if $F(u) \in L^t(\mathbb{R}^2)$ for $t > 1$ given by

$$\frac{2}{t} + \frac{2-\alpha}{2} = 2.$$

This implies that we must require

$$F(u) \in L^{\frac{4}{2+\alpha}}(\mathbb{R}^2).$$

We also need the following inequality, which will be used in the following lemmas.

$$(e^s - 1)^t \leq e^{ts} - 1, \quad \text{for } t > 1 \text{ and } s \geq 0.$$

Lemma 2.3. *Assume that $\{u_n\} \subset \widetilde{S}(a)$ is bounded and satisfies*

$$\limsup_{n\to\infty} |\nabla u_n|_2^2 \in \left(0, \frac{(2+\alpha)\pi}{\gamma} - a^2\right).$$

Then for $\gamma > \gamma_0$ close to γ_0, the sequence $\{e^{\gamma|u_n|^2} - 1\}$ is bounded in $L^t(\mathbb{R}^2)$ provided $t > 1$ close to 1.

Proof. Write

$$\beta := \limsup_{n\to\infty} ||u_n||_2^2.$$

Under the assumptions, we have that $\beta \in \left(0, \frac{(2+\alpha)\pi}{\gamma_0}\right)$. Then we can find some $\eta > 0$ such that $\beta < \frac{(2+\alpha)\pi}{\gamma_0+\eta}$. Without loss of generality, we may assume that

$$||u_n||_2^2 < \frac{(2+\alpha)\pi}{\gamma_0 + \eta}, \quad \forall n \in \mathbb{N}.$$

Since $\gamma > \gamma_0$ close to γ_0, we can write it as $\gamma = \gamma_0 + \xi$. Letting $t \in (1, 1+\xi)$ with $\xi \in \left(0, min\left\{\frac{\eta}{\gamma_0+2}, 1\right\}\right)$, we have

$$\limsup_{n\to\infty} t\gamma ||u_n||_2^2 \leq \limsup_{n\to\infty} (1+\xi)(\gamma_0+\xi) ||u_n||_2^2 \leq (\eta + \gamma_0)\beta < (2+\alpha)\pi < 4\pi.$$

Noting that $\beta > 0$, and by Lemma2.1, we arrive at

$$\begin{aligned}\limsup_{n\to\infty} \int_{\mathbb{R}^2} (e^{\gamma|u_n|^2} - 1)^t dx &\leq \limsup_{n\to\infty} \int_{\mathbb{R}^2} (e^{t\gamma|u_n|^2} - 1) dx \\ &= \limsup_{n\to\infty} \int_{\mathbb{R}^2} (e^{t\gamma||u_n||^2 \left(\frac{|u_n|}{||u_n||}\right)^2} - 1) dx \\ &< +\infty.\end{aligned}$$

The lemma is finished. □

Corollary 2.1. *Assume that $u_n \rightharpoonup u_0$ weakly in $\widetilde{S}(a)$ and $\limsup\limits_{n\to\infty} |\nabla(u_n-u_0)|_2^2 < \frac{(2+\alpha)\pi}{\gamma} - a^2$. Then for $\gamma > \gamma_0$ close to γ_0, we have that $\{e^{\gamma|u_n|^2}-1\}$ is bounded in $L^t(\mathbb{R}^2)$ provided $t>1$ close to 1.*

Proof. We just need to prove that when n large enough, for $\gamma > \gamma_0$ and $t>1$ close to 1, it still holds

$$\limsup_{n\to\infty} t\gamma||u_n-u_0||_2^2 < 4\pi. \tag{2.3}$$

Let $\upsilon_n = u_n - u_0$. By choosing $\omega > 0$ small enough, there exists some $C_\omega > 0$ such that

$$\begin{aligned}
\int_{\mathbb{R}^2} (e^{\gamma|u_n|^2}-1)^t dx &\le \int_{\mathbb{R}^2} (e^{t\gamma|u_n|^2}-1)dx \\
&= \int_{\mathbb{R}^2} (e^{t\gamma|\upsilon_n+u_0|^2}-1)dx \\
&\le \int_{\mathbb{R}^2} (e^{(1+\omega)t\gamma|\upsilon_n|^2+C_\omega t\gamma|u_0|^2}-1)dx. \\
&= \int_{\mathbb{R}^2} (e^{(1+\omega)t\gamma|\upsilon_n|^2}-1)(e^{C_\omega t\gamma|u_0|^2}-1)dx + \int_{\mathbb{R}^2} e^{(1+\omega)t\gamma|\upsilon_n|^2}dx + \int_{\mathbb{R}^2} e^{C_\omega t\gamma|u_0|^2}dx - 2 \\
&:= I + II + III + IV.
\end{aligned}$$

By choosing $r>1$ close to 1, according to (2.3), Lemma2.1 and the Hölder inequality, there exists some $C>0$ independent of n such that

$$\begin{aligned}
I &:= \int_{\mathbb{R}^2} (e^{(1+\omega)t\gamma|\upsilon_n|^2}-1)(e^{C_\omega t\gamma|u_0|^2}-1)dx \\
&\le \left(\int_{\mathbb{R}^2} (e^{(1+\omega)t\gamma|\upsilon_n|^2}-1)^r dx\right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^2} (e^{C_\omega t\gamma|u_0|^2}-1))^{r'} dx\right)^{\frac{1}{r'}} \\
&\le C,
\end{aligned}$$

$$II := \int_{\mathbb{R}^2} e^{(1+\omega)t\gamma|\upsilon_n|^2}dx \le C,$$

$$III := \int_{\mathbb{R}^2} e^{C_\omega t\gamma|u_0|^2}dx \le C,$$

where $r' = \frac{r}{r-1}$. Hence, $\{e^{\gamma|u_n|^2}-1\}$ is bounded in $L^t(\mathbb{R}^2)$ □

Lemma 2.4. *Assume that $\{u_n\} \subset \widetilde{S}(a)$ such that $u_n \rightharpoonup u_0$ weakly in $H^{1,2}_{rad}(\mathbb{R}^2)\bigcap \widetilde{S}(a)$ and*

$$\limsup_{n\to\infty} |\nabla u|_2^2 < \frac{2+\alpha}{\gamma} - a^2. \tag{2.4}$$

then we arrive at

$$|u_n|^q(e^{\gamma|u_n|^2}-1) \to |u|^q(e^{\gamma|u_0|^2}-1) \quad in \quad L^t(\mathbb{R}^2)$$

Proof. Setting
$$h_n(x) = e^{\gamma|u_n|^2-1}.$$
By (2.4) and Lemma 2.3, we have $\{h_n\}$ is a bounded sequence in $L^t(\mathbb{R}^2)$. By $u_n \rightharpoonup u_0$ in $H^{1,2}_{rad}(\mathbb{R}^2)$, we know that $u_n \to u$ a.e. in $\mathbb{R}^2$. Thus, we obtain $h_n(x) = e^{\gamma|u_n|^2-1} \to e^{\gamma|u|^2-1}$ a.e. in $\mathbb{R}^2$. Then, we have
$$h_n \rightharpoonup h = e^{\gamma|u|^2-1} \quad in \quad L^t(\mathbb{R}^2). \tag{2.5}$$
Now we show that
$$|u_n|^q \to |u|^q \quad in \quad L^{t'}(\mathbb{R}^2), \tag{2.6}$$
where $t' = \frac{t}{t-1}$. Then by the embedding $H^{1,2}_{rad}(\mathbb{R}^2) \hookrightarrow L^{qt'}(\mathbb{R}^2)$ is compact, we have
$$u_n \to u \quad L^{qt'}(\mathbb{R}^2).$$
Thus, we get (2.6). Together (2.5) with (2.6), we know
$$|u_n|^q(e^{\gamma|u_n|^2} - 1) \to |u|^q(e^{\gamma|u|^2} - 1) in \quad L^1(\mathbb{R}^2).$$
Then, the proof is complete.

□

Corollary 2.2. *Assume that* $(f_1)-(f_3)$ *hold, let* $\{u_n\} \subset \widetilde{S}(a) \cap H^{1,2}_{rad}(\mathbb{R}^2)$ *with*
$$\limsup_{n\to\infty} |\nabla u_n|_2^2 < \frac{(2+\alpha)\pi}{\gamma} - a^2.$$
If $u_n \rightharpoonup u$ *in* $H^{1,2}_{rad}(\mathbb{R}^2)$ *and* $u_n(x) \to u(x)$ *a.e. in* $\mathbb{R}$, *then*
$$\int_{\mathbb{R}^2} (I_\alpha * F(u_n))\, f(u_n)\phi dx \to \int_{\mathbb{R}^2} (I_\alpha * F(u))\, f(u)\phi dx,\ as\, n \to \infty,$$
for any $\phi \in C_0^\infty(\mathbb{R}^2)$.

Proof. First, we claim that $I_\alpha * F(u_n)$ belongs to $L^\infty(\mathbb{R}^2)$, indeed, by (2.2), we have
$$|F(u_n)| \le \varepsilon|u_n|^{\tau+1} + K_\varepsilon|u_n|^q(e^{\gamma|u_n|^2} - 1).$$
Then
$$\begin{aligned}|I_\alpha * F(u_n)| &= \left|\int_{\mathbb{R}^2} \frac{A_\alpha}{|x-y|^{2-\alpha}} F(u_n)dx\right| \\ &= \left|\int_{|x-y|\le1} \frac{A_\alpha}{|x-y|^{2-\alpha}} F(u_n)dx\right| + K_\varepsilon\left|\int_{|x-y|\ge1} \frac{A_\alpha}{|x-y|^{2-\alpha}} F(u_n)dx\right| \\ &= \int_{|x-y|\le1} \frac{A_\alpha\varepsilon|u_n|^{\tau+1}}{|x-y|^{2-\alpha}}dx + \int_{|x-y|\le1} \frac{A_\alpha K_\varepsilon|u_n|^q(e^{\gamma|u_n|^2}-1)}{|x-y|^{2-\alpha}}dx + K_\varepsilon\int_{|x-y|\ge1} \frac{A_\alpha\varepsilon|u_n|^{\tau+1}}{(x-y)^{2-\alpha}}dx \\ &+ \int_{|x-y|\ge1} K_\varepsilon|u_n|^q(e^{\gamma|u_n|^2} - 1)dx \\ &:= I + II + III + IV.\end{aligned}$$

Choose $\sigma \in (\frac{2}{\alpha}, \frac{4}{2+\alpha})$, by the Hölder inequality, we get

$$\begin{aligned} I &:= \int_{|x-y|\leq 1} \frac{A_\alpha \varepsilon |u_n|^{\tau+1}}{|x-y|^{2-\alpha}} dx \\ &\leq A_\alpha \varepsilon \left(\int_{|x-y|\leq 1} |u_n|^{(\tau+1)\sigma} dx \right)^{\frac{1}{\sigma}} \left(\int_{|x-y|\leq 1} \frac{1}{|x-y|^{(2-\alpha)\sigma\prime}} dx \right)^{\frac{1}{\sigma\prime}} \\ &< C_1. \end{aligned}$$

$$\begin{aligned} II &:= \int_{|x-y|\leq 1} \frac{A_\alpha K_\varepsilon |u_n|^q (e^{\gamma |u_n|^2} - 1)}{|x-y|^{2-\alpha}} dx \\ &\leq C A_\alpha K_\varepsilon \left(\int_{|x-y|\leq 1} |u_n|^{q\sigma t'} dx \right)^{\frac{1}{t'\sigma}} \left(\int_{|x-y|\leq 1} (e^{\gamma \sigma t |u_n|^2} - 1) dx \right)^{\frac{1}{t}} \\ &< C_2. \end{aligned}$$

Choose $\delta \to 0^+, st: q_{1,\delta} = \frac{(\tau+1)(2+\delta)}{\delta+\alpha} > 2$, and $t > 1$ close to 1, by the Hölder inequality, we get

$$\begin{aligned} III := \int_{|x-y|\geq 1} \frac{A_\alpha \varepsilon |u_n|^{\tau+1}}{(x-y)^{2-\alpha}} dx &\leq A_\alpha \varepsilon \left(\int_{|x-y|\leq 1} \frac{1}{|x-y|^{2+\delta}} dx \right)^{\frac{2-\alpha}{2+\delta}} \left(\int_{|x-y|\leq 1} |u_n|^{q_{1,\delta}} dx \right)^{\frac{\delta+\alpha}{\tau+\delta}} \\ &< C_3, \end{aligned}$$

$$\begin{aligned} IV := \int_{|x-y|\geq 1} K_\varepsilon |u_n|^q (e^{\gamma |u_n|^2} - 1) dx &\leq K_\varepsilon \left(\int_{|x-y|\leq 1} |u_n|^{qt'} dx \right)^{\frac{1}{t'}} \left(\int_{|x-y|\leq 1} (e^{t\gamma |u_n|^2} - 1) dx \right)^{\frac{1}{t}} \\ &< C_4, \end{aligned}$$

where $\sigma' = \frac{\sigma}{\sigma} - 1$, $t' = \frac{t}{t-1}$. This prove the claim.

Hence, for any $\phi \in C_0^\infty(\mathbb{R}^2)$, we have

$$|(I_\alpha * F(u_n)) f(u_n) \phi| \leq C \, |f(u_n)| \, |\phi| \leq \varepsilon |u_n|^\tau |\phi| + C |u_n|^{q-1} |\phi| (e^{\gamma |u_n|^2} - 1).$$

Let $U = supp\phi$. Then, by Lemma2.4, we obtain

$$\int_U |u_n|^\tau |\phi| dx \to \int_U |u|^\tau |\phi| dx, \quad as\, n \to \infty,$$

and

$$\int_U |u_n|^{q-1} |\phi| (e^{\gamma |u_n|^2} - 1) dx \to \int_U |u|^{q-1} |\phi| (e^{\gamma |u|^2} - 1) dx, \quad as\, n \to \infty,$$

Now, applying a variant of the Lebesgue dominated convergence theorem, we can deduce that

$$\int_{\mathbb{R}^2} (I_\alpha * F(u_n)) f(u_n) \phi dx \to \int_{\mathbb{R}^2} (I_\alpha * F(u)) f(u) \phi dx, \quad as\, n \to \infty.$$

which completes the proof. □

Corollary 2.3. *Assume that* $(f_1)-(f_3)$ *hold, let* $\{u_n\} \subset \widetilde{S}(a)\cap H^{1,2}_{rad}(\mathbb{R}^2)$ *with*

$$\limsup_{n\to\infty} |\nabla u_n|_2^2 < \frac{(2+\alpha)\pi}{\gamma} - a^2.$$

If $u_n \rightharpoonup u$ *in* $H^{1,2}_{rad}(\mathbb{R}^2)$ *and* $u_n(x)\to u(x)$ *a.e. in* $\mathbb{R}$, *then*

$$\int_{\mathbb{R}^2} (I_\alpha * F(u_n))\, F(u_n) dx \to \int_{\mathbb{R}^2} (I_\alpha * F(u))\, F(u) dx,$$

and

$$\int_{\mathbb{R}^2} (I_\alpha * F(u_n))\, f(u_n)u_n\ dx \to \int_{\mathbb{R}^2} (I_\alpha * F(u))\, f(u) u dx.$$

Proof. From Corollary2.2, we know

$$|I_\alpha * F(u_n)| \leqslant C.$$

By (2.2), we have

$$|F(u_n)| \le \varepsilon |u_n|^{\tau+1} + K_\varepsilon |u_n|^q (e^{\gamma|u_n|^2}-1).$$

where $\gamma>\gamma_0, \tau>2+\frac{2}{\alpha}, q>1+\frac{\alpha}{2}$. Hence, we have

$$|(I_\alpha * F(u_n))F(u_n)| \le C|F(u_n)| \le \varepsilon |u_n|^{\tau+1} + C|u_n|^q (e^{\gamma|u_n|^2}-1).$$

By the compact embedding $H^{1,2}_{rad}(\mathbb{R}^2)\hookrightarrow L^p(\mathbb{R}^2)$ for $p>2$, we have

$$u_n \to u \quad in \quad L^p(\mathbb{R}^2).$$

Now, applying a variant of the Lesbesgue dominated convergence theorem, we can deduce that

$$\int_{\mathbb{R}^2} (I_\alpha * F(u_n))\, F(u_n) dx \to \int_{\mathbb{R}^2} (I_\alpha * F(u))\, F(u) dx, \quad as\, n\to\infty.$$

A similar argument works to show that

$$\int_{\mathbb{R}^2} (I_\alpha * F(u_n))\, f(u_n)u_n\ dx \to \int_{\mathbb{R}^2} (I_\alpha * F(u))\, f(u) u dx, \quad as\, n\to\infty.$$

□

2.2 Perturbation setting

In order to recover the differentiability, we define for $\eta \in (0,1]$,

$$\begin{aligned} I_\eta(u) &:= \frac{\eta}{\theta}\int_{\mathbb{R}^2} |\nabla u|^\theta dx + I(u) \\ &= \frac{1}{2}\int_{\mathbb{R}^2}|\nabla u|^2 dx + \int_{\mathbb{R}^2}|u|^2|\nabla u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^2}(I_\alpha * F(u))\, F(u) dx \end{aligned} \tag{2.7}$$

on the space $O := H^{1,\theta}(\mathbb{R}^2)\bigcap H^{1,2}(\mathbb{R}^2)$, for some fixed θ, satisfying $2<\theta<3$. Then O is a reflexive Banach space, and by the Hardy-Littlewood-Sobolev inequality and [16, Lemma A.1], we can know that $I_\eta \in C^1(O)$. We will consider I_η on the constraint

$$S(a) := \left\{ u \in O : \int_{\mathbb{R}^2} |u|^2 dx = a^2 \right\}. \tag{2.8}$$

Recalling the L^2-norm preserved transform [15] $\mathcal{H}: H^{1,2}(\mathbb{R}^2)\times\mathbb{R}\to\mathbb{R}$ and

$$\mathcal{H}(u,s)(x)=e^s u(e^s x).$$

Then, we have

$$\begin{aligned} I_\eta(\mathcal{H}(u,s)) := {}& \frac{\eta}{\theta}e^{2(\theta-1)s}\int_{\mathbb{R}^2}|\nabla u|^\theta dx + e^{4s}\int_{\mathbb{R}^2}|u|^2|\nabla u|^2dx+\frac{1}{2}e^{2s}\int_{\mathbb{R}^2}|\nabla u|^2dx \\ & -\frac{1}{2e^{(2+\alpha)s}}\int_{\mathbb{R}^2}\left(I_\alpha * F(e^s u)\right)F(e^s u)dx. \end{aligned} \tag{2.9}$$

We define

$$\begin{aligned} P_\eta(u) :={}& \frac{d}{ds}|_{s=0} I_\eta(\mathcal{H}(u,s)) \\ ={}& \eta\frac{2(\theta-1)}{\theta}\int_{\mathbb{R}^2}|\nabla u|^\theta dx + 4\int_{\mathbb{R}^2}|u|^2|\nabla u|^2dx+\int_{\mathbb{R}^2}|\nabla u|^2dx+\frac{2+\alpha}{2}\int_{\mathbb{R}^2}\left(I_\alpha * F(u)\right)F(u)dx \quad (2.10)\\ & -\int_{\mathbb{R}^2}\left(I_\alpha * F(u)\right)f(u)udx, \quad (2.11) \end{aligned}$$

and $P_\eta\in C^1(O)$, then we define a manifold

$$\mathcal{P}_\eta(a) := \left\{u\in\mathcal{S}(a): P_\eta(u)=0\right\}. \tag{2.12}$$

We have the following results.

Lemma 2.5. *Any critical point u of $I_\eta|_{\mathcal{S}(a)}$ is contained in $\mathcal{P}_\eta(a)$.*

Proof. By [6, Lemma 3], there exists a $\lambda\in\mathbb{R}$ such that

$$I_\eta'(u)+\lambda u = 0 \quad \text{in} \quad O^*. \tag{2.13}$$

On the one hand, using (2.13), we obtain

$$\eta\int_{\mathbb{R}^2}|\nabla u|^\theta dx+4\int_{\mathbb{R}^2}|u|^2|\nabla u|^2dx+\int_{\mathbb{R}^2}|\nabla u|^2dx+\lambda\int_{\mathbb{R}^2}|u|^2dx-2\int_{\mathbb{R}^2}(I_\alpha * F(u))f(u)udx=0. \tag{2.14}$$

On the other hand, testing (2.13), for more details see [5, Proposition 1], we obtain

$$\eta\frac{\theta-2}{2\theta}\int_{\mathbb{R}^2}|\nabla u|^\theta dx-\frac{1}{2}\lambda\int_{\mathbb{R}^2}|u|^2dx+\frac{2+\alpha}{4}\int_{\mathbb{R}^2}(I_\alpha * F(u))F(u)dx+\frac{1}{2}\int_{\mathbb{R}^2}(I_\alpha * F(u))f(u)u=0. \tag{2.15}$$

Combining (2.14) and (2.15), we can get $P_\eta(u)=0$, so $u\in\mathcal{P}_\eta(a)$. □

Lemma 2.6. *The following statements hold: if $\sup_{n\geq1} I_\eta(u_n)<+\infty$ for $u_n\in\mathcal{P}_\eta(a)$, then*

$$\sup_{n\geq1} max\left\{\eta\int_{\mathbb{R}^2}|\nabla u_n|^\theta dx, \int_{\mathbb{R}^2}|u_n|^2|\nabla u_n|^2dx, \int_{\mathbb{R}^2}|\nabla u_n|^2dx\right\}<+\infty.$$

Proof. For any $u \in \mathcal{P}_\eta(a)$, exists $z \in \mathbb{R}$ with $\frac{1}{2k-2-\alpha} < z < \frac{1}{4}$, $k > 3 + \frac{2}{\alpha}$, by (f_3) we have

$$
\begin{aligned}
I_\eta(u) &= I_\eta(u) - zP_\eta(u)\\
&= \eta(\frac{1-2(\theta-1)z}{\theta})\int_{\mathbb{R}^2}|\nabla u|^\theta dx + (1-4z)\int_{\mathbb{R}^2}|u|^2|\nabla u|^2dx + (\frac{1-2z}{2})\int_{\mathbb{R}^2}|\nabla u|^2dx\\
&- \frac{(2+\alpha)z+1}{2}\int_{\mathbb{R}^2}(I_\alpha * F(u))F(u)dx + z\int_{\mathbb{R}^2}(I_\alpha * F(u))F(u)dx\\
&\geq \eta(\frac{1-2(\theta-1)z}{\theta})\int_{\mathbb{R}^2}|\nabla u|^\theta dx + (1-4z)\int_{\mathbb{R}^2}|u|^2|\nabla u|^2dx - (\frac{(2+\alpha)z+1}{2})\int_{\mathbb{R}^2}(I_\alpha * F(u))F(u)dx\\
&+ kz\int_{\mathbb{R}^2}(I_\alpha * F(u))F(u)dx\\
&= \eta(\frac{1-2(\theta-1)z}{\theta})\int_{\mathbb{R}^2}|\nabla u|^\theta dx + (1-4z)\int_{\mathbb{R}^2}|u|^2|\nabla u|^2dx + (\frac{1-2z}{2})\int_{\mathbb{R}^2}|\nabla u|^2dx\\
&+ \left(kz - (\frac{(2+\alpha)z+1}{2})\right)\int_{\mathbb{R}^2}(I_\alpha * F(u))F(u)dx
\end{aligned}
$$

As for $\frac{1-(2\theta-2)z}{\theta} > 0$, $1-4z > 0$, $\frac{1-2z}{2} > 0$, $kz - (\frac{(2+\alpha)z+1}{2}) > 0$, so the conclusion has finished. □

3 The minimax approach

In this section, we will prove that $I_\eta(\mathcal{H}(u,s))$ on $S(a)\times\mathbb{R}$ possesses a kind of mountain-pass geometrical structure.

Lemma 3.1. *For any $0 < \eta \leq 1$ and $u \in S(a)$ be arbitrary but fixed, the following statements hold:*
(i) $|\nabla\mathcal{H}(u,s)|_2 \to 0^+$ and $I_\eta(\mathcal{H}(u,s)) \to 0^+$ as $s \to -\infty$;
(ii) $|\nabla\mathcal{H}(u,s)|_2 \to +\infty$ and $I_\eta(\mathcal{H}(u,s)) \to -\infty$ as $s \to +\infty$.

Proof. By a straightforward calculation, it follows that

$$
\int_{\mathbb{R}^2}|\mathcal{H}(u,s)(x)|^2dx = a^2,\quad \int_{\mathbb{R}^2}|\mathcal{H}(u,s)(x)|^\varsigma dx = e^{(\varsigma-2)s}\int_{\mathbb{R}^2}|u(x)|^\varsigma dx \quad \text{for all } \varsigma > 2,
$$

$$
\int_{\mathbb{R}^2}|\nabla\mathcal{H}(u,s)(x)|^2dx = e^{2s}\int_{\mathbb{R}^2}|\nabla u(x)|^2dx,\quad \int_{\mathbb{R}^2}|\nabla\mathcal{H}(u,s)(x)|^\theta dx = e^{(2\theta-2)s}\int_{\mathbb{R}^2}|\nabla u(x)|^\theta dx,
$$

$$
\int_{\mathbb{R}^2}|\mathcal{H}(u,s)(x)|^2|\nabla\mathcal{H}(u,s)(x)|^2dx = e^{4s}\int_{\mathbb{R}^2}|u(x)|^2|\nabla u(x)|^2dx.
$$

From the above equalities, fixing $\varsigma > 2$, as $s \to -\infty$, it follows that

$$
|\mathcal{H}(u,s)|_\varsigma^\varsigma \to 0^+,\quad |\nabla\mathcal{H}(u,s)|_2^2 \to 0^+,\quad |\nabla\mathcal{H}(u,s)|_\theta^\theta \to 0^+,\quad \int_{\mathbb{R}^2}|\mathcal{H}(u,s)|^2|\nabla\mathcal{H}(u,s)|^2dx \to 0^+.
$$

By (2.2), we have

$$
|F(\mathcal{H}(u,s))| \leq \varepsilon|\mathcal{H}(u,s)|^{\tau+1} + K_{\varepsilon,\mu}|\mathcal{H}(u,s)|^q(e^{\gamma|\mathcal{H}(u,s)|^2} - 1), \tag{3.1}
$$

For all $\gamma||\mathcal{H}(u,s)||^2 < (2+\alpha)\pi$, by Lemma2.1, we have

$$
\int_{\mathbb{R}^2}\left(e^{\gamma|\mathcal{H}(u,s)|^2} - 1\right)dx = \int_{\mathbb{R}^2}\left(e^{\gamma||\mathcal{H}(u,s)||^2\left(\frac{|\mathcal{H}(u,s)|}{||\mathcal{H}(u,s)||}\right)^2} - 1\right)dx \leq C.
$$

Hence, using Hardy-Littlewood-Sobolev inequality, Minkowski inequality and the Hölder′s inequality, we deduce that

$$\begin{aligned}\left(\int_{\mathbb{R}^2}|F(\mathcal{H}(u,s))|^{\frac{4}{2+\alpha}}dx\right)^{\frac{2+\alpha}{4}} &\leq \left(\int_{\mathbb{R}^2}[\varepsilon|\mathcal{H}(u,s)|^{\tau+1}dx+K_{\varepsilon,\mu}|\mathcal{H}(u,s)|^{q}(e^{\gamma|\mathcal{H}(u,s)|^2}-1)]^{\frac{4}{2+\alpha}}dx\right)^{\frac{2+\alpha}{4}}\\ &\leq \left(\int_{\mathbb{R}^2}\varepsilon|\mathcal{H}(u,s)|^{\frac{4(\tau+1)}{2+\alpha}}dx\right)^{\frac{2+\alpha}{4}}\\ &+K_{\varepsilon,\mu}\left(\int_{\mathbb{R}^2}|\mathcal{H}(u,s)|^{\frac{4qt'}{2+\alpha}}dx\right)^{\frac{2+\alpha}{4t'}}\left(\int_{\mathbb{R}^2}e^{\frac{4t\gamma}{2+\alpha}|\mathcal{H}(u,s)|^2}-1dx\right)^{\frac{2+\alpha}{4t}},\end{aligned}$$

where $t,t'>1$ satisfying $\frac{1}{t}+\frac{1}{t'}=1$. Then, there exists $t>1$ close to 1 such that

$$t\gamma||\mathcal{H}(u,s)||^2<(2+\alpha)\pi,$$

which implies that

$$\left(\int_{\mathbb{R}^2}e^{\frac{4t\gamma}{2+\alpha}|\mathcal{H}(u,s)|^2}-1dx\right)^{\frac{2+\alpha}{4t}}\leq C. \tag{3.2}$$

Note that, by (3.2) and the Hölder′s inequality, we arrive at

$$\begin{aligned}\int_{\mathbb{R}^2}(I_\alpha*F(\mathcal{H}(u,s)))\,F(\mathcal{H}(u,s)dx &\leq |F(\mathcal{H}(u,s))|_{\frac{4}{2+\alpha}}\,|F(\mathcal{H}(u,s))|_{\frac{4}{2+\alpha}}\\ &\leq\left(\varepsilon\,|\mathcal{H}(u,s)|^{\tau+1}_{\frac{4(\tau+1)}{2+\alpha}}+C\,|\mathcal{H}(u,s)|^{q}_{\frac{4qt'}{2+\alpha}}\right)^2.\end{aligned}$$

Thus, we conclude that

$$\int_{\mathbb{R}^2}(I_\alpha*F(\mathcal{H}(u,s)))\,F(\mathcal{H}(u,s)dx\leq\left(\varepsilon e^{\frac{2\tau-\alpha}{2}s}|u|^{\tau+1}_{\frac{4(\tau+1)}{2+\alpha}}+Ce^{\frac{4qt'-4-2\alpha}{4t'}s}|u|^{q}_{\frac{4qt'}{2+\alpha}}\right)^2.$$

And then, we have

$$\begin{aligned}I_\eta(\mathcal{H}(u,s)) &\geq \frac{\eta}{\theta}e^{2(\theta-1)s}\int_{\mathbb{R}^2}|\nabla u|^\theta dx+e^{4s}\int_{\mathbb{R}^2}|u|^2|\nabla u|^2dx+\frac{1}{2}e^{2s}\int_{\mathbb{R}^2}|\nabla u|^2dx\\ &-\frac{1}{2e^{(2+\alpha)s}}\int_{\mathbb{R}^2}(I_\alpha*F(e^su))F(e^su)dx\\ &\geq \frac{\eta}{\theta}e^{2(\theta-1)s}\int_{\mathbb{R}^2}|\nabla u|^\theta dx+e^{4s}\int_{\mathbb{R}^2}|u|^2|\nabla u|^2dx+\frac{1}{2}e^{2s}\int_{\mathbb{R}^2}|\nabla u|^2dx\\ &-\left(\varepsilon e^{\frac{2\tau-\alpha}{2}s}|u|^{\tau+1}_{\frac{4(\tau+1)}{2+\alpha}}+Ce^{\frac{(4qt'-4-2\alpha)s}{4t'}}|u|^{q}_{\frac{4qt'}{2+\alpha}}\right)^2.\\ &:=I_{\eta,1}(\mathcal{H}(u,s))\end{aligned}$$

Thus, by $\tau>2+\frac{\alpha}{2}, q>1+\frac{\alpha}{2}$, we know that

$$I_{\eta,1}(\mathcal{H}(u,s))\to 0^+\quad\text{as}\quad s\to-\infty,$$

and

$$I_{\eta,1}(\mathcal{H}(u,s)) \to -\infty \quad \text{as} \quad s \to +\infty.$$

On the other hand, we define

$$g(z) = \int_{\mathbb{R}^2} (I_\alpha * F(z))F(z)dx,$$

by (2.9),

$$\begin{aligned} I_\eta(\mathcal{H}(u,s)) := & \frac{\eta}{\theta} e^{2(\theta-1)s} \int_{\mathbb{R}^2} |\nabla u|^\theta dx + e^{4s} \int_{\mathbb{R}^2} |u|^2 |\nabla u|^2 dx + \frac{1}{2} e^{2s} \int_{\mathbb{R}^2} |\nabla u|^2 dx \\ & - \frac{1}{2e^{(2+\alpha)s}} \int_{\mathbb{R}^2} (I_\alpha * F(e^s u))\, F(e^s u) dx. \end{aligned}$$

Set

$$w(t) = g(\frac{tu}{||u||}),$$

where $t = e^s$. By $f(_3)$, we know

$$\frac{w'(t)}{w(t)} \geq \frac{2k}{t}$$

then, by integral operation, we obtain

$$g(tu) \geq g(\frac{u}{||u||}) s^{2k} ||u||^{2k}.$$

Therefore, we have

$$\begin{aligned} I_\eta(\mathcal{H}(u,s)) & \leq C_5 e^{2s} + C_6 e^{2(\theta-1)s} + C_7 e^{4s} - C_8 e^{(2k-(2+\alpha))s} \\ & := I_{\eta,2}(\mathcal{H}(u,s)) \end{aligned}$$

Thus, by $2 < 2(\theta-1) < 4 < 2k - (2+\alpha)$, we know that

$$I_{\eta,2}(\mathcal{H}(u,s)) \to 0^+ \quad \text{as} \quad s \to -\infty,$$

and

$$I_{\eta,2}(\mathcal{H}(u,s)) \to -\infty \quad \text{as} \quad s \to +\infty.$$

Moreover, the inequality below also yields that

$$I_\eta(\mathcal{H}(u,s)) \to 0^+ \quad \text{as} \quad s \to -\infty,$$

$$I_\eta(\mathcal{H}(u,s)) \to -\infty \quad \text{as} \quad s \to +\infty.$$

□

To recover the compactness, we shall study I_η on the radial space:

$$S_r(a) := S(a) \cap O_r, \quad O_r := H^{1,\theta}_{rad}(\mathbb{R}^2) \cap H^{1,2}_{rad}(\mathbb{R}^2).$$

Lemma 3.2. *There exists $K(a,\mu)>0$ small enough such that*

$$0<\sup_{u\in\mathcal{A}} I_\eta(u)<\inf_{u\in\mathcal{B}} I_\eta(u) \tag{3.3}$$

with

$$\mathcal{A}=\Big\{u\in S_r(a),\eta\int_{\mathbb{R}^2}|\nabla u|^\theta dx+\int_{\mathbb{R}^2}|u|^2|\nabla u|^2dx+\int_{\mathbb{R}^2}|\nabla u|^2dx\le K(a,\mu)\Big\},$$

and

$$\mathcal{B}=\Big\{u\in S_r(a),\eta\int_{\mathbb{R}^2}|\nabla u|^\theta dx+\int_{\mathbb{R}^2}|u|^2|\nabla u|^2dx+\int_{\mathbb{R}^2}|\nabla u|^2dx=3K(a,\mu)\Big\}.$$

Moreover, $K(a,\mu)\to 0$ when $\mu\to\infty$.

Proof. Firstly, for $u\in\mathcal{A}$, and $v\in\mathcal{B}$, we have the following estimations

$$\begin{aligned}&\frac{\eta}{\theta}\int_{\mathbb{R}^2}|\nabla u|^\theta dx+\int_{\mathbb{R}^2}|u|^2|\nabla u|^2dx+\frac{1}{2}\int_{\mathbb{R}^2}|\nabla u|^2dx\\ \le&\max\left\{\frac{1}{\theta},\frac{1}{2},1\right\}\Big(\eta\int_{\mathbb{R}^2}|\nabla u|^\theta dx+\int_{\mathbb{R}^2}|u|^2|\nabla u|^2dx+\int_{\mathbb{R}^2}|\nabla u|^2dx\Big)\\ \le&K(a,\mu),\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}&\frac{\eta}{\theta}\int_{\mathbb{R}^2}|\nabla v|^\theta dx+\int_{\mathbb{R}^2}|v|^2|\nabla v|^2dx+\frac{1}{2}\int_{\mathbb{R}^2}|\nabla v|^2dx\\ \ge&\min\left\{\frac{1}{\theta},\frac{1}{2},1\right\}\Big(\eta\int_{\mathbb{R}^2}|\nabla v|^\theta dx+\int_{\mathbb{R}^2}|v|^2|\nabla v|^2dx+\int_{\mathbb{R}^2}|\nabla v|^2dx\Big)\\ =&\frac{3}{\theta}K(a,\mu).\end{aligned} \tag{3.5}$$

Now, let $K(a,\mu)<\frac{(2+\alpha)\pi}{3\gamma}-\frac{a^2}{3}$. Thus, we have

$$||v||^2=|\nabla v|_2^2+|v|_2^2<3K(a,\mu)+a^2<\frac{(2+\alpha)\pi}{\gamma}.$$

Then, similar as Lemma3.1, we obatin

$$\begin{aligned}\int_{\mathbb{R}^2}(I_\alpha*F(v))F(v)dx\le&\Big(\varepsilon|v|^{\tau+1}_{\frac{4(\tau+1)}{2+\alpha}}+C|v|^q_{\frac{4qt'}{2+\alpha}}\Big)^2\\ \le&\varepsilon|v|^{2(\tau+1)}_{\frac{4(\tau+1)}{2+\alpha}}+C|v|^{2q}_{\frac{4qt'}{2+\alpha}}\end{aligned}$$

where $\tau>2+\frac{\alpha}{2},q>1+\frac{\alpha}{2}$. By the Gagliardo-Sobolev inequality(1.6), we have

$$\int_{\mathbb{R}^2}(I_\alpha*F(v))F(v)dx\le C|\nabla v|_2^{2\tau-\alpha}a^{\frac{2+\alpha}{2}}+C|\nabla v|_2^{\frac{2qt'-2-\alpha}{t'}}a^{\frac{2+\alpha}{2t'}}.$$

From (f_3), we have $(I_\alpha*F(u))F(u)>0$ for any $u\in H^{1,2}(\mathbb{R})$, then, we have

$$I_\eta(v)-I_\eta(u)\ge\Big(\frac{\eta}{\theta}\int_{\mathbb{R}^2}|\nabla v|^\theta dx+\int_{\mathbb{R}^2}|v|^2|\nabla v|^2dx+\frac{1}{2}\int_{\mathbb{R}^2}|\nabla v|^2dx\Big)$$

$$
\begin{aligned}
&-\Big(\frac{\eta}{\theta}\int_{\mathbb{R}^2}|\nabla u|^\theta dx+\int_{\mathbb{R}^2}|u|^2|\nabla u|^2dx+\frac{1}{2}\int_{\mathbb{R}^2}|\nabla u|^2dx\Big)-\frac{1}{2}\int_{\mathbb{R}^2}(I_\alpha*F(v))F(v)dx\\
&\geq \frac{3}{\theta}K(a,\mu)-K(a,\mu)-C\left(K(a,\mu)\right)^{\frac{2\tau-\alpha}{2}}a^{\frac{2+\alpha}{2}}-C\left(K(a,\mu)\right)^{\frac{2qt'-2-\alpha}{2t'}}a^{\frac{2+\alpha}{2t'}}\\
&=\frac{3-\theta}{\theta}K(a,\mu)-C\left(K(a,\mu)\right)^{1+\frac{2\tau-(2+\alpha)}{2}}-C\left(K(a,\mu)\right)^{1+\frac{2qt'-2-\alpha-2}{2t'}}
\end{aligned}
$$

Since $\tau>2+\frac{\alpha}{2}$, $2<\theta<3$ and $t'>0$ with $\frac{2qt'-2-\alpha}{2t'}>1$ and $\frac{3-\theta}{\theta}>0$, fixing

$$
K(a,\mu)=min\left\{\frac{(2+\alpha)\pi}{3\gamma_0}-\frac{a^2}{3},\ \left[\frac{3-\theta}{C\theta}\right]^{\frac{2}{2\tau-(2+\alpha)}},\ \left[\frac{3-\theta}{C\theta}\right]^{\frac{2t'}{2qt'-2-2t'-\alpha}}\right\}, \tag{3.6}
$$

and so,

$$
I_\eta(v)-I_\eta(u)\geq\frac{3-\theta}{2\theta}K(a,\mu)>0,
$$

which shows the desired result. Finally, in order to prove the limit $K(a,\mu)\to 0$ when $\mu\to\infty$, fix $u_0\in S_r(a)$ with $\eta\int_{\mathbb{R}^2}|\nabla u_0|^\theta dx+\int_{\mathbb{R}^2}|u_0|^2|\nabla u_0|^2dx+\int_{\mathbb{R}^2}|\nabla u_0|^2dx\leq K(a,\mu)$. Then, (f_4) together with Lemma3.1 ensure that

$$
\frac{\mu^2}{2}\int_{\mathbb{R}^2}(I_\alpha*|u_0|^\sigma)|u_0|^\sigma dx\leq\int_{\mathbb{R}^2}(I_\alpha*F(u_0))F(u_0)dx\leq C|u_0|^{2(\tau+1)}_{\frac{4(\tau+1)}{2+\alpha}}+C|u_0|^{2q}_{\frac{4qt'}{2+\alpha}}.
$$

Therefore, we must have $C\to\infty$ when $\mu\to\infty$, and so $C\to\infty$ when $\mu\to\infty$. This limit together with (3.6) show that $K(a,\mu)\to 0$ when $\mu\to\infty$. □

Similar discussion as the last lemma, we have the following Corollary.

Corollary 3.1. For $K(a,\mu)>0$ given in (3.6), there holds that $I_\eta(u)>0$, for all $u\in S_r(a)$ with $\eta\int_{\mathbb{R}^2}|\nabla u|^\theta dx+\int_{\mathbb{R}^2}|u|^2|\nabla u|^2dx+\int_{\mathbb{R}^2}|\nabla u|^2dx\leq K(a,\mu)$. Moreover,

$$
I_\eta^{**}=inf\left\{I_\eta(u):u\in S_r(a)\text{ and }\eta\int_{\mathbb{R}^2}|\nabla u|^\theta dx+\int_{\mathbb{R}^2}|u|^2|\nabla u|^2dx+\int_{\mathbb{R}^2}|\nabla u|^2dx=\frac{K(a,\mu)}{3}\right\}>0.
$$

Proof. By Lemma3.1 and the Gagliardo-Nirenberg-type inequality, we have

$$
\begin{aligned}
I_\eta(u)&=\frac{\eta}{\theta}\int_{\mathbb{R}^2}|\nabla u|^\theta dx+\int_{\mathbb{R}^2}|u|^2|\nabla u|^2dx+\frac{1}{2}\int_{\mathbb{R}^2}|\nabla u|^2dx-\frac{1}{2}\int_{\mathbb{R}^2}(I_\alpha*F(u))F(u)dx\\
&\geq\frac{\eta}{\theta}\int_{\mathbb{R}^2}|\nabla u|^\theta dx+\int_{\mathbb{R}^2}|u|^2|\nabla u|^2dx+\frac{1}{2}\int_{\mathbb{R}^2}|\nabla u|^2dx-C\left(\int_{\mathbb{R}^2}|u|^2|\nabla u|^2dx\right)^{\frac{2\tau-\alpha}{4}}\\
&\quad-C\left(\int_{\mathbb{R}^2}|u|^2|\nabla u|^2dx\right)^{\frac{2qt'-2-\alpha}{4t'}},
\end{aligned}
$$

where $\frac{2\tau-\alpha}{4}>1$ and $\frac{2qt'-2-\alpha}{2t'}>1$. For any $u\in\partial\mathcal{A}(K(a,\mu),a)$,

$$
\partial\mathcal{A}(K(a,\mu),a):=\left\{u\in S_r(a):\eta\int_{\mathbb{R}^2}|\nabla u|^\theta dx+\int_{\mathbb{R}^2}|u|^2|\nabla u|^2dx+\int_{\mathbb{R}^2}|\nabla u|^2dx=K(a,\mu)\right\}.
$$

For a smaller $0 < \rho < K(a,\mu)$, we can get

$$
\begin{aligned}
\inf_{\partial\mathcal{A}(\rho,a)} I_\eta(u) &\geq \frac{\eta}{\theta}\int_{\mathbb{R}^2}|\nabla u|^\theta dx + \int_{\mathbb{R}^2}|u|^2|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^2}|\nabla u|^2 dx - C\left(\int_{\mathbb{R}^2}|u|^2|\nabla u|^2 dx\right)^{\frac{2\tau-\alpha}{4}} \\
&\quad - C\left(\int_{\mathbb{R}^2}|u|^2|\nabla u|^2 dx\right)^{\frac{2qt'-2-\alpha}{4t'}}, \\
&\geq \frac{\eta}{\theta}\int_{\mathbb{R}^2}|\nabla u|^\theta dx + \frac{1}{2}\int_{\mathbb{R}^2}|\nabla u|^2 dx + C\int_{\mathbb{R}^2}|u|^2|\nabla u|^2 dx \\
&\geq C\rho > 0,
\end{aligned}
$$

for $K(a,\mu) > 0$ small enough. the proof is completed. □

In what follows, we fix $u_0 \in S_r(a)$ and apply Lemma 3.1, Lemma 3.2 and Corollary 3.1 to get two numbers $s_1 = s_1(u_0,a,\mu) < 0$, and $s_2 = s_2(u_0,a,\mu) > 0$, the functions $u_{1,\mu} = \mathcal{H}(u_0,s_1)$ and $u_{2,\mu} = \mathcal{H}(u_0,s_2)$ satisfy

$$
\eta\int_{\mathbb{R}^2}|\nabla u_{1,\mu}|^\theta dx + \int_{\mathbb{R}^2}|u_{1,\mu}|^2|\nabla u_{1,\mu}|^2 dx + \int_{\mathbb{R}^2}|\nabla u_{1,\mu}|^2 dx < \frac{K(a,\mu)}{3} \quad \textit{with} \quad I_\eta(u_{1,\mu}) > 0,
$$

and

$$
\eta\int_{\mathbb{R}^2}|\nabla u_{2,\mu}|^\theta dx + \int_{\mathbb{R}^2}|u_{2,\mu}|^2|\nabla u_{2,\mu}|^2 dx + \int_{\mathbb{R}^2}|\nabla u_{2,\mu}|^2 dx > 3K(a,\mu) \quad \textit{with} \quad I_\eta(u_{2,\mu}) < 0.
$$

Now, following the idea from Jeanjean [15], we fix the following mountain pass level given by

$$
\gamma_\mu(a) := \inf_{h\in\Gamma}\max_{t\in[0,1]} I_\eta(h(t)),
$$

where

$$
\begin{aligned}
\Gamma = \{h \in C([0,1],S_r(a)) : \eta\int_{\mathbb{R}^2}|\nabla h(0)|^\theta dx + \int_{\mathbb{R}^2}|h(0)|^2|\nabla h(0)|^2 dx + \int_{\mathbb{R}^2}|\nabla h(0)|^2 dx < \frac{K(a,\mu)}{3}, \\
I_\eta(h(1)) < 0\}.
\end{aligned}
$$

From Corollary 3.1, there exists $t_0 \in (0,1)$ such that

$$
\max_{t\in[0,1]} I_\eta(h(t)) \geq I_\eta(h(t_0)) \geq I_\eta^{**} > 0,
$$

where I_η^{**} was given in Corollary 3.1. Then we obtain that

$$
\gamma_\mu(a) \geq I_\eta^{**} > 0.
$$

Lemma 3.3. *There holds* $\lim_{\mu\to+\infty}\gamma_\mu(a) = 0$.

Proof. In what follows, we set the path $h_0(t) = \mathcal{H}(u_0,(1-t)s_1 + ts_2) \in \Gamma$. Then, by (f_4),

$$
\begin{aligned}
\gamma_\mu(a) &\leq \max_{t\in[0,1]} I_\eta(h_0(t)) \\
&\leq \max_{t\in[0,1]}\left\{\frac{\eta}{\theta}\int_{\mathbb{R}^2}|\nabla h_0(t)|^\theta dx + \frac{1}{2}\int_{\mathbb{R}^2}|\nabla h_0(t)|^2 dx + \int_{\mathbb{R}^2}|h_0(t)|^2|\nabla h_0(t)|^2 dx\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\mu^2}{2}\int_{\mathbb{R}^2}\left(I_\alpha*|h_0(t)|^\sigma\right)|h_0(t)|^\sigma dx\Big\}\\
&=\max_{t\in[0,1]}\Big\{\frac{\eta}{\theta}r^{2\theta-2}\int_{\mathbb{R}^2}|\nabla u_0|^\theta dx+\frac{1}{2}r^2\int_{\mathbb{R}^2}|\nabla u_0|^2dx+r^4\int_{\mathbb{R}^2}|u_0|^2|\nabla u_0|^2dx\\
&-\frac{\mu^2}{2}r^{2\sigma-(2+\alpha)}\int_{\mathbb{R}^2}\left(I_\alpha*|u_0|^\sigma\right)|u_0|^\sigma dx\Big\}\\
&:=\max_{t\in[0,1]}g_0(r),
\end{aligned}
$$

where $r:=e^{(1-t)s_1+ts_2}$. **Case 1**. If $0<r<1$, we have

$$
\begin{aligned}
g_0(r)\le&\, r^2 max\left\{\frac{1}{\theta},\frac{1}{2},1\right\}\left(\eta\int_{\mathbb{R}^2}|\nabla u_0|^\theta dx+\int_{\mathbb{R}^2}|\nabla u_0|^2dx+\int_{\mathbb{R}^2}|u_0|^2|\nabla u_0|^2dx\right)\\
&-\frac{\mu^2}{2}r^{2\sigma-(2+\alpha)}\int_{\mathbb{R}^2}\left(I_\alpha*|u_0|^\sigma\right)|u_0|^\sigma dx\\
=&r^2\left(\eta\int_{\mathbb{R}^2}|\nabla u_0|^\theta dx+\int_{\mathbb{R}^2}|\nabla u_0|^2dx+\int_{\mathbb{R}^2}|u_0|^2|\nabla u_0|^2dx\right)\\
&-\frac{\mu^2}{2}r^{2\sigma-(2+\alpha)}\int_{\mathbb{R}^2}\left(I_\alpha*|u_0|^\sigma\right)|u_0|^\sigma dx\\
&:=g_1(r).
\end{aligned}
$$

It is not difficult to check that g_1 has a unique critical point $\widetilde{r}$ on $(0,+\infty)$, which is a global maximum point at positive level.

$$
\widetilde{r}=\left[\frac{4\left(\eta\int_{\mathbb{R}^2}|\nabla u_0|^\theta dx+\int_{\mathbb{R}^2}|\nabla u_0|^2dx+\int_{\mathbb{R}^2}|u_0|^2|\nabla u_0|^2dx\right)}{\mu^2(2\sigma-(2+\alpha))\int_{\mathbb{R}^2}\left(I_\alpha*|u_0|^\sigma\right)|u_0|^\sigma dx}\right]^{\frac{1}{2\sigma-4-\alpha}}>0,
$$

and so,

$$
\gamma_\mu(a)\le C\left(\frac{1}{\mu}\right)^{\frac{2}{2\sigma-4-\alpha}}\to 0,\quad \text{as}\quad \mu\to+\infty.
$$

Case 2. If $r\ge 1$, we have

$$
\begin{aligned}
g_0(r)\le&\, r^4\left(\eta\int_{\mathbb{R}^2}|\nabla u_0|^\theta dx+\int_{\mathbb{R}^2}|\nabla u_0|^2dx+\int_{\mathbb{R}^2}|u_0|^2|\nabla u_0|^2dx\right)\\
&-\frac{\mu^2}{2}r^{2\sigma-(2+\alpha)}\int_{\mathbb{R}^2}\left(I_\alpha*|u_0|^\sigma\right)|u_0|^\sigma dx.
\end{aligned}
$$

Discussed as before, we have

$$
\gamma_\mu(a)\le C\left(\frac{1}{\mu}\right)^{\frac{2}{2\sigma-6-\alpha}}\to 0,\quad \text{as}\quad \mu\to+\infty.
$$

Hence,

$$
\gamma_\mu(a)\le min\left\{C\left(\frac{1}{\mu}\right)^{\frac{2}{2\sigma-4-\alpha}},C\left(\frac{1}{\mu}\right)^{\frac{2}{2\sigma-6-\alpha}}\right\}\to 0,\quad \text{as}\quad \mu\to+\infty,
$$

for some $C>0$ (possibly different) that do not depend on $\mu>0$. □

To find a Palais-Smale sequence, we consider an auxilary functional

$$\widetilde{I}_\eta(s,u) := I_\eta(\mathcal{H}(u,s)) = I_\eta(h(t)) : \mathbb{R}\times S_r(a) \to \mathbb{R}. \tag{3.7}$$

Notice that $\widetilde{I}_\eta$ is of class C^1, by the symmetric critical point principle [25], a Palais-Smale sequence for $\widetilde{I}_\eta|_{\mathbb{R}\times S_r(a)}$ is also a Palais-Smale sequence for $\widetilde{I}_\eta|_{\mathbb{R}\times S(a)}$. Denoting the closed sublevel set by

$$I_\eta^c = \{u \in S(a) : I_\eta(u) \le c\},$$

we also define

$$\sigma_\eta(a) := \inf_{\widetilde{h}\in\Gamma_\eta}\max_{t\in[0,1]} \widetilde{I}_\eta(\widetilde{h}(t)),$$

where

$$\Gamma_\eta := \Big\{\widetilde{h} = (\gamma,\beta) \in C([0,1],\mathbb{R}\times S_r(a)) : \eta\int_{\mathbb{R}^2}|\nabla\beta(0)|^\theta dx + \int_{\mathbb{R}^2}|\beta(0)|^2|\nabla\beta(0)|^2dx + \int_{\mathbb{R}^2}|\nabla\beta(0)|^2dx < \frac{K(a,\mu)}{3}, I_\eta(\beta(1)) < 0, \gamma(0)=0, \gamma(1)=0\Big\},$$

Obviously, it holds that $\sigma_\eta(a) = \gamma_\mu(a)$.

The same discussed as in [16, Lemma 3.6], taking a minimizing sequence $\{\widetilde{h}_n = (0,\beta_n)\} \subset \Gamma_\eta$ with $\beta_n \ge 0$ a.e. in $\mathbb{R}^2$, there exists a Palais-Smale sequence $\{(s_n,w_n)\} \subset \mathbb{R}\times S_r(a)$ for $\widetilde{I}_\eta|_{\mathbb{R}\times S_r(a)}$ at level $\sigma_\eta(a)$. Let $u_n = \mathcal{H}(w_n,s_n)$, we have

$$\begin{aligned} -\Delta u_n - u_n\Delta u_n^2 + \lambda_n u_n = (I_\alpha * F(u_n))F(u_n) + o_n(1), \quad in\ O^*, \\ I_\eta(u_n) \to \sigma_\eta(a) = \gamma_\mu(a), \quad \text{as} \quad n\to+\infty, \end{aligned} \tag{3.8}$$

with the additional property that

$$|s_n| + dist_O(w_n,\beta_n([0,1])) \to 0, \quad \text{as} \quad n\to\infty,$$

Moreover, for some sequence $\{\lambda_n\}\subset\mathbb{R}$, and

$$\begin{aligned} P_\eta(u_n) &= \eta\frac{2(\theta-1)}{\theta}\int_{\mathbb{R}^2}|\nabla u_n|^\theta dx + 4\int_{\mathbb{R}^2}|u_n|^2|\nabla u_n|^2dx + \int_{\mathbb{R}^2}|\nabla u_n|^2dx \\ &+ \frac{2+\alpha}{2}\int_{\mathbb{R}^2}(I_\alpha * F(u_n))F(u_n)dx - \int_{\mathbb{R}^2}(I_\alpha * F(u_n))f(u_n)u_ndx \\ &\to 0, \quad \text{as} \quad n\to+\infty. \end{aligned} \tag{3.9}$$

From Lemma 2.6, we know that $\{u_n\}$ is bounded in O_r, and so, the number λ_n must satisfy the equality below

$$\lambda_n = \frac{1}{a^2}\left\{-\eta\int_{\mathbb{R}^2}|\nabla u_n|^\theta dx - 4\int_{\mathbb{R}^2}|u_n|^2|\nabla u_n|^2dx - \int_{\mathbb{R}^2}|\nabla u_n|^2dx + \int_{\mathbb{R}^2}(I_\alpha * F(u_n))f(u_n)u_ndx\right\} + o_n(1).$$

Lemma 3.4. *There holds*

$$\limsup_{n\to+\infty}\int_{\mathbb{R}^2}(I_\alpha * F(u_n))F(u_n)dx \le \frac{4}{k-3-\frac{\alpha}{2}}\gamma_\mu(a).$$

Proof. Using the fact that $I_\eta(u_n)=\gamma_\mu(a)+o_n(1)$ and $P_\eta(u_n)=o_n(1)$, it follows that

$$\begin{aligned}&\eta\frac{2\theta+\alpha}{\theta}\int_{\mathbb{R}^2}|\nabla u_n|^\theta dx+(6+\alpha)\int_{\mathbb{R}^2}|u_n|^2|\nabla u_n|^2dx+\frac{4+\alpha}{2}\int_{\mathbb{R}^2}|\nabla u_n|^2dx-\int_{\mathbb{R}^2}(I_\alpha*F(u_n))F(u_n)dx\\&=(2+\alpha)\gamma_\mu(a)+o_n(1).\end{aligned}$$

Moreover, by $I_\eta(u_n)=\gamma_\mu(a)+o_n(1)$, we have

$$\begin{aligned}&\frac{6+\alpha}{2}\int_{\mathbb{R}^2}(I_\alpha*F(u_n))F(u_n)dx+(6+\alpha)\gamma_\mu(a)+o_n(1)\\&=\eta\frac{6+\alpha}{\theta}\int_{\mathbb{R}^2}|\nabla u_n|^\theta dx+(6+\alpha)\int_{\mathbb{R}^2}|u_n|^2|\nabla u_n|^2dx+\frac{6+\alpha}{2}\int_{\mathbb{R}^2}|\nabla u_n|^2dx\\&\ge\eta\frac{2(\theta+\alpha)}{\theta}\int_{\mathbb{R}^2}|\nabla u_n|^\theta dx+(6+\alpha)\int_{\mathbb{R}^2}|u_n|^2|\nabla u_n|^2dx+\frac{4+\alpha}{2}\int_{\mathbb{R}^2}|\nabla u_n|^2dx\\&=\int_{\mathbb{R}^2}(I_\alpha*F(u_n))f(u_n)u_ndx+(2+\alpha)\gamma_\mu(a)+o_n(1).\end{aligned}$$

As $k>3+\frac{\alpha}{2}, 2<\theta<3$ and (f_3), we have that

$$\limsup_{n\to+\infty}\int_{\mathbb{R}^2}(I_\alpha*F(u_n))F(u_n)dx\le\frac{4}{k-(3+\frac{\alpha}{2})}\gamma_\mu(a).$$

□

Lemma 3.5. *The sequence* $\{u_n\}$ *satisfies*

$$\limsup_{n\to+\infty}\left(\eta\int_{\mathbb{R}^2}|\nabla u_n|^\theta dx+\int_{\mathbb{R}^2}|u_n|^2|\nabla u_n|^2dx+\int_{\mathbb{R}^2}|\nabla u_n|^2dx\right)\le\theta(\frac{k-1-\frac{\alpha}{2}}{k-3-\frac{\alpha}{2}})\gamma_\mu(a).$$

Hence, there exists $\mu^*>0$ *such that*

$$\limsup_{n\to+\infty}\left(\eta\int_{\mathbb{R}^2}|\nabla u_n|^\theta dx+\int_{\mathbb{R}^2}|u_n|^2|\nabla u_n|^2dx+\int_{\mathbb{R}^2}|\nabla u_n|^2dx\right)<\frac{(2+\alpha)\pi}{\gamma_0}-a^2\quad for\quad\forall\mu\ge\mu^*.$$

Proof. Since $I_\eta(u_n)=\gamma_\mu(a)+o_n(1)$, we have

$$\begin{aligned}&\limsup_{n\to+\infty}\int_{\mathbb{R}^2}(I_\alpha*F(u_n))F(u_n)dx+2\gamma_\mu(a)+o_n(1)\\&=\eta\frac{2}{\theta}\int_{\mathbb{R}^2}|\nabla u_n|^\theta dx+2\int_{\mathbb{R}^2}|u_n|^2|\nabla u_n|^2dx+\int_{\mathbb{R}^2}|\nabla u_n|^2dx\\&>\frac{2}{\theta}\left(\eta\int_{\mathbb{R}^2}|\nabla u_n|^\theta dx+\int_{\mathbb{R}^2}|u_n|^2|\nabla u_n|^2dx+\int_{\mathbb{R}^2}|\nabla u_n|^2dx\right).\end{aligned}$$

Therefore, by lemma 3.4, we have

$$\limsup_{n\to+\infty}\left(\eta\int_{\mathbb{R}^2}|\nabla u_n|^\theta dx+\int_{\mathbb{R}^2}|u_n|^2|\nabla u_n|^2dx+\int_{\mathbb{R}^2}|\nabla u_n|^2dx\right)\le\theta(\frac{k-1-\frac{\alpha}{2}}{k-3-\frac{\alpha}{2}})\gamma_\mu(a).$$

By lemma 3.3, we can directly obtain the second inequality, the prove is complete. □

Lemma 3.6. *Fix $\mu \geq \mu^*$, where μ^* is given in Lemma 3.5. Then, $\{\lambda_n\}$ is a bounded sequence with*

$$\limsup_{n\to+\infty} |\lambda_n| \leq \left(\frac{8\theta(k-1-\frac{\alpha}{2}) + 2(2+\alpha)}{a^2(k-3-\frac{\alpha}{2})} + \right) \gamma_\mu(a)$$

and

$$\liminf_{n\to+\infty} \lambda_n > \frac{2+\alpha}{2a^2} \liminf_{n\to+\infty} \int_{\mathbb{R}^2} (I_\alpha * F(u_n))F(u_n)dx.$$

Proof. From Lemma 2.6 and Lemma 3.5, we know that $\{u_n\}$ is bounded in O_r, and the boundedness of $\{u_n\}$ yields that $\{\lambda_n\}$ is bounded, indeed

$$\lambda_n |u_n|_2^2 = -\eta \int_{\mathbb{R}^2} |\nabla u_n|^\theta dx - 4\int_{\mathbb{R}^2} |u_n|^2 |\nabla u_n|^2 dx - \int_{\mathbb{R}^2} |\nabla u_n|^2 dx + \int_{\mathbb{R}^2} (I_\alpha * F(u_n))f(u_n)u_n dx + o_n(1),$$

as for $|u_n|_2^2 = a^2$, we have

$$\lambda_n a^2 = -\eta \int_{\mathbb{R}^2} |\nabla u_n|^\theta dx - 4\int_{\mathbb{R}^2} |u_n|^2 |\nabla u_n|^2 dx - \int_{\mathbb{R}^2} |\nabla u_n|^2 dx + \int_{\mathbb{R}^2} (I_\alpha * F(u_n))f(u_n)u_n dx + o_n(1). \tag{3.10}$$

Hence,

$$|\lambda_n| a^2 \leq \eta \int_{\mathbb{R}^2} |\nabla u_n|^\theta dx + 4\int_{\mathbb{R}^2} |u_n|^2 |\nabla u_n|^2 dx + \int_{\mathbb{R}^2} |\nabla u_n|^2 dx + \int_{\mathbb{R}^2} (I_\alpha * F(u_n))f(u_n)u_n dx + o_n(1).$$

The limit (3.9) together with Lemma 3.4 and Lemma 3.5 ensure that the sequence $\left\{ \int_{\mathbb{R}^2} (I_\alpha * F(u_n))f(u_n)u_n dx \right\}$ is bounded, because

$$\begin{aligned}
&\limsup_{n\to+\infty} \int_{\mathbb{R}^2} (I_\alpha * F(u_n))f(u_n)u_n dx \\
&< \limsup_{n\to+\infty} \left[4\left(\eta \int_{\mathbb{R}^2} |\nabla u_n|^\theta dx + \int_{\mathbb{R}^2} |u_n|^2 |\nabla u_n|^2 dx + \int_{\mathbb{R}^2} |\nabla u_n|^2 dx \right) + \frac{2+\alpha}{2} \int_{\mathbb{R}^2} (I_\alpha * F(u_n))F(u_n)dx \right] \\
&< \frac{8\theta(k-1-\frac{\alpha}{2}) + 2(2+\alpha)}{k-3-\frac{\alpha}{2}} \gamma_\mu(a).
\end{aligned}$$

which concludes that $\{\lambda_n\}$ is a bounded sequence with

$$\limsup_{n\to+\infty} |\lambda_n| \leq \left(\frac{8\theta(k-1-\frac{\alpha}{2}) + 2(2+\alpha)}{a^2(k-3-\frac{\alpha}{2})} + \right) \gamma_\mu(a)$$

In order to proof the second equality, the equality (3.10) together with the limit (3.9) lead to

$$\begin{aligned}
\lambda_n a^2 &= -\eta \int_{\mathbb{R}^2} |\nabla u_n|^\theta dx - 4\int_{\mathbb{R}^2} |u_n|^2 |\nabla u_n|^2 dx - \int_{\mathbb{R}^2} |\nabla u_n|^2 dx + \int_{\mathbb{R}^2} (I_\alpha * F(u_n))f(u_n)u_n dx \\
&\quad + o_n(1) \\
&> -\eta \frac{2(\theta-1)}{\theta} \int_{\mathbb{R}^2} |\nabla u_n|^\theta dx - 4\int_{\mathbb{R}^2} |u_n|^2 |\nabla u_n|^2 dx - \int_{\mathbb{R}^2} |\nabla u_n|^2 dx + \int_{\mathbb{R}^2} (I_\alpha * F(u_n))f(u_n)u_n dx \\
&\quad + o_n(1) \\
&= \frac{2+\alpha}{2} \int_{\mathbb{R}^2} (I_\alpha * F(u_n))F(u_n)dx + o_n(1),
\end{aligned}$$

showing the desired result. □

For fixed $\eta \in (0,1]$ and $\forall \mu \geq \mu^*$, from lemma 3.5, we can conclude that

$$\limsup_{n\to+\infty} |\nabla u_n|_2^2 < \frac{(2+\alpha)\pi}{\gamma_0} - a^2.$$

According to Corollary 2.3, we deduce that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^2} (I_\alpha * F(u_n))f(u_n)u_n dx = \int_{\mathbb{R}^2} (I_\alpha * F(u_\eta))f(u_\eta)u_\eta dx$$

and

$$\lim_{n\to+\infty} \int_{\mathbb{R}^2} (I_\alpha * F(u_n))F(u_n) dx = \int_{\mathbb{R}^2} (I_\alpha * F(u_\eta))F(u_\eta) dx,$$

where $u_n \rightharpoonup u_\eta$ in $H_r^{1,2}(\mathbb{R}^2)$. The last limit implies that $u_\eta \neq 0$, because otherwise, by Corollary 2.3 and the limit equation (3.9), we have

$$\lim_{n\to+\infty} \int_{\mathbb{R}^2} (I_\alpha * F(u_n))F(u_n) dx = \lim_{n\to+\infty} \int_{\mathbb{R}^2} (I_\alpha * F(u_n))f(u_n)u_n dx = 0,$$

and by lemma 3.6, we derive that

$$\liminf_{n\to+\infty} \lambda_n \geq \frac{2+\alpha}{2a^2} \int_{\mathbb{R}^2} (I_\alpha * F(u_n))F(u_n) dx = 0.$$

Thus, we have $\lambda_n \geq 0$. Since $\{u_n\}$ is bounded in $H_r^{1,2}(\mathbb{R}^2)$. Corollary 2.3 together with (f_1) and (f_2) and the following equality

$$\begin{aligned}\lambda_n |u_n|_2^2 = &-\eta \int_{\mathbb{R}^2} |\nabla u_n|^\theta dx - 4\int_{\mathbb{R}^2} |u_n|^2 |\nabla u_n|^2 dx - \int_{\mathbb{R}^2} |\nabla u_n|^2 dx + \int_{\mathbb{R}^2} (I_\alpha * F(u_n))f(u_n)u_n dx \\ &+ o_n(1),\end{aligned}$$

leads to

$$\lambda_n a^2 = -\eta \int_{\mathbb{R}^2} |\nabla u_n|^\theta dx - 4\int_{\mathbb{R}^2} |u_n|^2 |\nabla u_n|^2 dx - \int_{\mathbb{R}^2} |\nabla u_n|^2 dx + o_n(1).$$

From this, one has

$$\begin{aligned} 0 \geq -\liminf_{n\to+\infty} \lambda_n a^2 &= \limsup_{n\to+\infty} (-\lambda_n) a^2 \\ &= \limsup_{n\to+\infty} \left(\eta \int_{\mathbb{R}^2} |\nabla u_n|^\theta dx + 4\int_{\mathbb{R}^2} |u_n|^2 |\nabla u_n|^2 dx + \int_{\mathbb{R}^2} |\nabla u_n|^2 dx \right) \\ &\geq \liminf_{n\to+\infty} \left(\eta \int_{\mathbb{R}^2} |\nabla u_n|^\theta dx + 4\int_{\mathbb{R}^2} |u_n|^2 |\nabla u_n|^2 dx + \int_{\mathbb{R}^2} |\nabla u_n|^2 dx \right) \\ &\geq 0, \end{aligned}$$

then, we obtain that

$$\eta \int_{\mathbb{R}^2} |\nabla u_n|^\theta dx + 4\int_{\mathbb{R}^2} |u_n|^2 |\nabla u_n|^2 dx + \int_{\mathbb{R}^2} |\nabla u_n|^2 dx \to 0,$$

and this is impossible, because $\gamma_\mu(a) > 0$.

4 Critical points of $I_\eta|_{S(a)}$

The above analysis ensure that the weak limit u of $\{u_n\}$ is nontrivial. Moreover, the equality

$$\liminf_{n\to+\infty} \lambda_n \geq \frac{2+\alpha}{2a^2} \liminf_{n\to+\infty} \int_{\mathbb{R}^2} (I_\alpha * F(u_n))F(u_n)dx$$

ensures that

$$\liminf_{n\to+\infty} \lambda_n = \frac{2+\alpha}{2a^2} \int_{\mathbb{R}^2} (I_\alpha * F(u_\eta))F(u_\eta)dx > 0.$$

From this, going up to subsequence, still denoted by $\{\lambda_n\}$, we can assume that

$$\lambda_n \to \lambda_\eta > 0, \quad \text{as} \quad n \to +\infty.$$

Since $\{u_n\}$ is bounded, we have

$$I_\eta'(u_n) + \lambda_\eta u_n \to 0 \quad in\ O^*.$$

Then, from [16, Lemma A.2], we have

$$I_\eta'(u_\eta) + \lambda_\eta u_\eta = 0, \tag{4.1}$$

testing (4.1) with $x \cdot \nabla u_\eta$ and u_η, we obtain $P_\eta(u_\eta) = 0$. It follows that

$$\begin{aligned} &P_\eta(u_n) + \int_{\mathbb{R}^2} (I_\alpha * F(u_n))f(u_n)u_n dx - \frac{2+\alpha}{2} \int_{\mathbb{R}^2} (I_\alpha * F(u_n))F(u_n)dx \\ &\to P_\eta(u_\eta) + \int_{\mathbb{R}^2} (I_\alpha * F(u_\eta))f(u_\eta)u_\eta dx - \frac{2+\alpha}{2} \int_{\mathbb{R}^2} (I_\alpha * F(u_\eta))F(u_\eta)dx. \end{aligned}$$

Then using the weak lower semicontinuous property, see [10, Lemma 4.3], there must be

$$\eta\frac{2(\theta-1)}{\theta} \int_{\mathbb{R}^2} |\nabla u_n|^\theta dx \to \eta\frac{2(\theta-1)}{\theta} \int_{\mathbb{R}^2} |\nabla u_\eta|^\theta dx, \tag{4.2}$$

$$\int_{\mathbb{R}^2} |u_n|^2|\nabla u_n|^2 dx \to \int_{\mathbb{R}^2} |u_\eta|^2|\nabla u_\eta|^2 dx, \tag{4.3}$$

$$\int_{\mathbb{R}^2} |\nabla u_n|^2 dx \to \int_{\mathbb{R}^2} |\nabla u_\eta|^2 dx. \tag{4.4}$$

That gives $I_\eta(u_\eta) = \lim_{n\to+\infty} I_\eta(u_n) = \gamma_\mu(a)$. Moreover, from (4.2)–(4.4), we obtain

$$I_\eta'(u_n)[u_n] \to I_\eta'(u_\eta)[u_\eta]. \tag{4.5}$$

Thus combining (4.5) with (4.1), there holds $\lambda_n|u_n|_2^2 \to \lambda_\eta|u_\eta|_2^2$. Since $\lambda_\eta > 0$, the last limit implies that $u_n \to u_\eta$ in O, implying that $|u_\eta|_2^2 = a^2$.

Based on the above preliminary works, we conclude that

Theorem 4.1. *For any fixed $\eta \in (0,1]$, there exists a $u_\eta \in O_r \setminus \{0\}$ and a $\lambda_\eta \in \mathbb{R}$ such that*

$$I'\eta(u_\eta) + \lambda_\eta u_\eta = 0,$$

$$I_\eta(u_\eta) = \gamma_\mu(a), \quad P_\eta(u_\eta) = 0,$$

$$|u_\eta|_2^2 = a^2.$$

5 Proof of Theorem 1.1

By Theorem 4.1, we can take

$$\eta_n \to 0^+,\ I'_{\eta_n}(u_{\eta_n}) - \lambda_{\eta_n} u_{\eta_n} = 0,\ \text{and}\ I_{\eta_n}(u_{\eta_n}) \to d^*(a) := \lim_{\eta_n \to 0^+} \gamma_\mu(a) \in (0, +\infty),$$

for $u_{\eta_n} \in S_r(a)$ with $|u_{\eta_n}|_2^2 = a^2$, then Lemma 3.6 implies that $\lambda_{\eta_n} < 0$. Now Theorem **??** gives that there exists $v \neq 0, v \in H^{1,2}_{rad}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, and $\lambda_0 \in \mathbb{R}$ such that

$$I'(v) - \lambda_0 v = 0, \quad I(v) = d^*(a), \quad \text{and} \quad |v|_2^2 = a^2.$$

That is, v is a nontrivial radial solution of (1.4).

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