UNDER PEER REVIEW

HANKEL DETERMINANT FOR A NEW SUBCLASS OF ANALYTIC FUNCTIONS INVOLVING CHEBYSHEV POLYNOMIALS AND TOEPLITZ DETERMINANT

ABSTRACT. In this paper, a new subclass $S^*(\mu, \beta, \wp, U_n(t))$ of univalent functions is defined by using Opoola differential operator involving subordination principle.The upper bounds to the second Hankel determinant denoted by $H_2(2)$ for the subclass $S^*(\mu, \beta, \wp, U_n(t))$ is established in connection with Chebyshev polynomials of the second kind and Toeplitz determinants

1. Introduction and preliminaries

Let A be the class of functions $f(z)$ defined by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Denote by S the subclass of A consisting of functions which are analytic, univalent in the unit disk $\mathbb{U}$ and normalized by $f(0) = 0 = f'(0) - 1$.
A function $f(z) \in S$ of the form 1.1 is star-like in the unit disk $\mathbb{U}$ if it maps a unit disk onto a star-like domain. A necessary and sufficient condition for a function $f(z)$ to be star-like is that

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0, (z \in \mathbb{U})$$

The class of all star-like functions is denoted by S^*.

An analytic function $f(z)$ of the form (1.1) is convex if it maps the unit disk $\mathbb{U}$ conformally onto a convex domain. Equivalently, a function $f(z)$ is said to be convex if and only if it satisfies the following condition;

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \ (z \in \mathbb{U}).$$

The class of all convex functions is denoted by K

Let $f(z)$ and $g(z)$ be analytic functions in the unit disk $\mathbb{U}$, then $f(z)$ is subordinate to $g(z)$ in the unit $\mathbb{U}$ written as $f(z) \prec g(z)$, if there exist a function $\omega(z)$ analytic in the unit $\mathbb{U}$ satisfying the conditions $\omega(0)$=0, $|\omega(z)| < 1$ which is called the Schwartz function such that $f(z) = g(\omega(z))$. If the function g is univalent in

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the unit $\mathbb{U}$, then $f(z) \prec g(z), z \in \mathbb{U} \Longleftrightarrow f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.
Let P be the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} c_n z^n \tag{1.2}$$

which are analytic in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. If $p(z) \in P$ satisfies the conditions $Re(p(z)) > 0$ and $p(0) = 1$, for $z \in \mathbb{U}$, then $p(z)$ is called a Carathéodory function or function having positive real part in the unit disk $\mathbb{U}$.

Definition 1.1. [17] For $\wp \geq 0$, $0 \leq \mu \leq \beta, n \in \mathbb{N}_0, z \in \mathbb{U}$, Opoola Differential Operator $D^n(\mu, \beta, \wp)f(z) : A \to A$ is defined as;

$$D^0(\mu, \beta, \wp)f(z) = f(z)$$
$$D^1(\mu, \beta, \wp)f(z) = \wp z f'(z) - z(\beta - \mu)\wp + (1 + (\beta - \mu - 1)\wp)f(z)$$
$$D^n(\mu, \beta, \wp)f(z) = (D(D^{n-1}(\mu, \beta, \wp)f(z)))$$

From the above definition and for $f(z)$ in Equation (1.1)

$$D^n(\mu, \beta, \wp)f(z) = z + \sum_{k=2}^{\infty} [1 + (k + \beta - \mu - 1)\wp]^n a_k z^k \tag{1.3}$$

Remark 1.2. The following are some remarks on Opoola Differential Operator.

(1) When $\wp = 1$, $\mu = \beta$, then $D^n(\mu, \beta, \wp)f(z)$ reduces to Sălăgean differential operator. [19]
(2) When $\mu = \beta$, then $D^n(\mu, \beta, \wp)f(z)$ reduces to Al-Oboudi differential operator. [1]

It can be seen from Remark 1.2 that Opoola differential operator is a generalization of Sălăgean differential operator and Al-Oboudi differential operator, some of the researchers that have studied Opoola differential operator and established certain geometric properties for the classes of functions introduced are, but not limited to [3, 4, 6, 7, 13, 14, 15, 16, 18, 20].

Definition 1.3. A function $f(z)$ given by Equation (1.1) is said to be in the class $S^*(\mu, \beta, \wp, U_n(t))$ if

$$\frac{z(D^n(\mu, \beta, \wp)f(z))'}{(D^n(\mu, \beta, \wp)f(z))} \prec H(z, t) \tag{1.4}$$

where $D^n(\mu, \beta, \wp)f(z)$ is the Opoola differential operator for $f(z)$ and $H(z, t)$ is the Chebyshev polynomials of the second kind for $t \in (\frac{1}{2}, 1]$.

Remark 1.4. When $n = 1, \beta = \mu, \wp = 1$, Equation (1.4) reduces to the class

$$\left(1 + \left(\frac{zf''(z)}{f'(z)}\right)\right) \prec H(z, t)$$

studied in [10].

Lemma 1.5. *[5] If $p \in P$ has the series expansion $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots, z \in \mathbb{U}$ with $Re(p(z)) > 0$ and $(p(0)) = 1$. Then $|p_n| \leq 2,\ n = \{1, 2, \cdots\}$.*

Lemma 1.6. *[8] Let the function $p \in P$ given by Equation (1.2) with $Re(p(z)) > 0, p(0) = 1$ and has the power series representation $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, then*
$2p_2 = p_1^2 + x(4 - p_1^2)$,
$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z$
for some x, y with $|x| \leq 1, |z| \leq 1$

1.1. **Chebyshev Polynomials.** The Chebyshev polynomials named after the Russian Mathematician Pafnuty Chebyshev form a special class of polynomials suitable for approximating other functions. They are widely used in many areas of mathematics like numerical solution of ordinary and partial differential equations, analytic functions just to mention few. The Chebyshev polynomial of the second kind is given by

$$H(z,t) = 1 + 2tz + (4t^2 - 1)z^2 + (8t^3 - 4t)z^3 + (16t^4 - 12t^2 + 1)z^4 + ... \quad (1.5)$$

where
$U_0(t) = 1, U_1(t) = 2t, U_2(t) = (4t^2 - 1), U_3(t) = (8t^3 - 4t), U_4(t) = (16t^4 - 12t^2 + 1)...$
for $t \in (0, 1], z \in \mathbb{U}$.

1.2. **Hankel determinant for class of functions $f(z) \in S$.** For $q \geq 1$ and $n \geq 1$, the q-th Hankel determinant for $f(z)$ is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & a_{n+2} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & a_{n+3} & \cdots & a_{n+q} \\ a_{n+2} & a_{n+3} & a_{n+4} & \cdots & a_{n+q+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & a_{n+q-1} & \cdots & a_{n+2q-2} \end{vmatrix} \quad (1.6)$$

Several authors have studied the Hankel determinant in order to examine it's growth rate as $n \longrightarrow \infty$ and to also establish it's bounds for diverse precise values of q and n. It is discovered recently that few of the researchers in geometric function theory have studied the bounds on Hankel determinant for different classes of univalent functions in collaboration with Chebyshev polynomials of the second kind. See [2, 9, 11, 12, 15].
For $q = 2$ and $n = 1, H_2(1) = |a_3 - \sigma a_2^2|$, is known as the Fekete-Szego functional for $\sigma = 1$.
For $q, n = 2$,

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}$$

known as the second Hankel determinant has been studied by many researchers and its upper bound estimates for various subclasses of univalent and multivalent analytic functions were discussed. Motivated by [10] and [11], next is to discuss the geometric properties of a new subclass $S^*(\mu, \beta, \wp, U_n(t))$ of univalent functions defined in Equation (1.4).

2. Main results

2.1. **Hankel Determinants for the Class $S^*(n, \mu, \beta, \wp, U_n(t))$.**

Theorem 2.1. *Let the function f(z) given by (1.1) be in the class* $S^*(n,\mu,\beta,\wp,U_n(t))$, *for* $t\in(\frac{1}{2},1]$ *then*

$$|a_2a_4-a_3^2|\leq\begin{cases} Q(2^-,t) if \chi_1,\chi_2\geq 0\\ \frac{t^2}{M_3^2} if \chi_1,\chi_2\leq 0\\ max\{\frac{t^2}{M_3^2},Q(2^-,t)\} if \chi_1>0,\chi_2<0\\ max\{Q(p_0,t),Q(2^-,t)\} if \chi_1<0,\chi_2>0\end{cases} \tag{2.1}$$

where;

$$\chi_1=\frac{2t}{3M_2M_4}\left\{28t^3+14t^2+5t-2\right\}+\frac{t}{3M_3^2}\left\{2t[4(t^2+t)-3]-1\right\}$$

$$\chi_2=\frac{2t^2}{3M_2M_4}\{17t^2+2t+2\}+\frac{t}{M_3^2}\{8t^2+t+1\}$$

$$Q(2^-,t)=\frac{t^2}{M_3^2}+\chi_1+\chi_2$$

$$Q(p_0^+,t)=\frac{t^2}{M_3^2}$$

Proof: Let $f\in S^*(n,\mu,\beta,\wp,U_n(t))$, then there's a Schwartz function $w(z)$, analytic in the unit disk $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1$ such that from the Definition of subordination and Equation (1.4) ;

$$\frac{z(D^n(\mu,\beta,\wp)f(z))'}{D^n(\mu,\beta,\wp)f(z)}=H(w(z),t) \tag{2.2}$$

from the Definition of Chebyshev polynomial,

$$H(z,t)=\frac{1}{1-2zt+z^2}$$

$$H(w(z),t)=\frac{1}{1-2t(w(z))+(w(z))^2}=\sum_{n=0}^{\infty}U_n(t)(w(z))^n$$

$$=1+U_1(t)(w(z))+U_2(t)(w(z))^2+U_3(t)(w(z))^3+U_4(t)(w(z))^4+...$$

$$=\frac{U_1(t)p_1z}{2}+\left[\frac{U_1(t)}{2}\left(p_2-\frac{p_1^2}{2}\right)+U_2(t)\frac{p_1^2}{4}\right]z^2$$
$$+\left[\frac{U_1(t)}{2}\left(p_3-p_1p_2+\frac{p_1^3}{4}\right)+\frac{U_2(t)}{2}\left(p_1p_2-\frac{p_1^3}{2}\right)+\frac{U_3(t)p_1^3}{8}\right]z^3$$
$$+\left[\frac{U_1(t)}{2}\left(p_4-p_1p_3-\frac{p_2^2}{2}+\frac{3p_1^2p_2}{4}-\frac{p_1^4}{8}\right)+\frac{U_2(t)}{2}\left(p_1c_3+\frac{p_2^2}{2}-\frac{3p_1^2p_2}{2}+\frac{3p_1^4}{8}\right).\right.$$
$$\left.+\frac{U_3(t)}{8}\left(3p_1^2p_2-\frac{3p_1^4}{2}\right)+\frac{U_4(t)p_1^4}{16}\right]z^4+\cdots \tag{2.3}$$

From Equation (1.3)
Let $[1+(k+\beta-\mu-1)\wp]^n = M_k^n$
then the Opoola differential operator can be written as
$D^n(\mu,\beta,\wp)f(z) = z + \sum_{k=2}^{\infty} M_k^n a_k z^k$
where

$$M_2^n = [1+(\beta-\mu+1)\wp]^n$$
$$M_3^n = [1+(\beta-\mu+2)\wp]^n$$
$$M_4^n = [1+(\beta-\mu+3)\wp]^n$$
$$M_5^n = [1+(\beta-\mu+4)\wp]^n$$

Also,

$$\frac{z(D^n(\mu,\beta,\wp)f(z))'}{D^n(\mu,\beta,\wp)f(z)} = \frac{1+\sum_{n=0}^{\infty} k[1+(k+\beta-\mu-1)\wp]^n a_k z^{k-1}}{1+\sum_{n=0}^{\infty} k[1+(k+\beta-\mu-1)\wp]^n a_k z^{k-1}}$$

Using binomimal expansion for the denominator and multiplying through by the numerator, then;

$$\frac{z(D^n(\mu,\beta,\wp)f(z))'}{D^n(\mu,\beta,\wp)f(z)} = 1+M_2a_2z+(2M_3a_3-M_2^2a_2^2)z^2+(3M_4a_4-3M_2M_3a_2a_3+2M_2^3a_2^3)z^3+... \tag{2.4}$$

comparing the coefficients of like powers of z, z^2, z^3 in (2.3) and (2.4), the following holds;

$$a_2 = \frac{U_1(t)tp_1}{M_2}$$

$$a_3 = \frac{1}{M_3}\left[\frac{U_1(t)tp_2}{4} - \frac{p_1^2}{8}\left(U_1(t) - U_2(t) - U_1(t)^2\right)\right]$$

$$a_4 = \frac{1}{M_4}\left[p_1^3\left\{\frac{U_1(t)}{24} - \frac{U_2(t)}{12} + \frac{U_3(t)}{24} + \frac{U_1^2(t)}{16} + \frac{U_1(t)U_2(t)}{16} - \frac{U_1^3(t)}{48}\right\} + p_1p_2\left\{\frac{U_2(t)}{6} + \frac{U_1(t)}{6} + \frac{U_1^2(t)}{8}\right\} + \frac{U_1(t)p_3}{6}\right]$$

Remark 2.2. [10]
If $f \in C(1,1,0,0,1,U_n(t)), t \in (\frac{1}{2}, 1]$ then;
$|a_2| \leq t$
$|a_3| \leq \frac{4t^2}{3} - \frac{1}{6}$

Next is to get the upper bounds to the second Hankel determinant for the class $S^*(n,\mu,\beta,\wp,U_n(t))$. From Equation (1.6), the second Hankel determinant of analytic functions $f(z)$ for $q=2, n=2; q,n \in \mathbb{N}$ is defined by

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2a_4 - a_3^2|$$

Thus,

$$|a_2a_4-a_3^2| = \left|p_1^4\left\{\frac{U_1^2(t)}{48M_2M_4}-\frac{U_1(t)U_2(t)}{24M_2M_4}+\frac{U_1(t)U_3(t)}{48M_2M_4}-\frac{U_1^3(t)}{32M_2M_4}+\frac{U_1^2(t)U_2(t)}{32M_2M_4}\right.\right.$$
$$-\frac{U_1^4(t)}{96M_2M_4}-\frac{U_1^2(t)}{64M_3^2}+\frac{U_1(t)U_2(t)}{32M_3^2}+\frac{U_1^3(t)}{32M_3^2}-\frac{U_1^2(t)U_2(t)}{32M_3^2}-\frac{U_2^2(t)}{64M_3^2}-\frac{U_1^4(t)}{64M_3^2}+p_1^2p_2\left\{\frac{U_1(t)U_2(t)}{12M_2M_4}\right.$$
$$-\frac{U_1^2(t))}{12M_2M_4}+\frac{U_1^2(t)}{16M_2M_4}+\frac{U_1^3(t)}{16M_2M_4}-\frac{U_1(t)U_2(t)}{16M_3^2}-\frac{U_1^3(t)}{16M_3^2}+\frac{U_{1^2}(t)}{16M_3^2}-\frac{U_1^2(t)p_2^2}{16M_3^2}+\frac{U_1^2(t)p_1p_3}{16M_2M_4} \tag{2.5}$$

By using lemma 1.5 and applying triangle inequality, Equation (2.5) yields

$$|a_2a_4-a_3^2| \leq p_1^4\left\{\frac{U_1(t)U_3(t)}{48M_2M_4}+\frac{U_1^2(t)}{32M_2M_4}+\frac{U_1^4(t)}{96M_2M_4}+\frac{U_1^3(t)U_2(t)}{32M_3^2}+\frac{U_2^2(t)}{64M_3^2}-\frac{U_1^4(t)}{64M_3^2}\right\}$$
$$+\frac{U_1^2(t)p_1^3}{24M_2M_4}+\frac{U_1(t)p_1}{6M_2M_4}+|x|(4-p_1^2)p_1^2\left\{\frac{U_1(t)U_2(t)}{24M_2M_4}-\frac{U_1^3(t))}{32M_2M_4}+\frac{U_1(t)U_3(t)}{48M_2M_4}+\frac{U_1^2(t)}{16M_3^2}+\frac{U_1(t)U_2(t)}{32M_3^2}\right\}$$
$$+|x|(2p_1^2-p_1^4)\frac{U_1^2(t)}{32M_3^2}+|x|^2(4-p_1^2)^2\frac{U_1(t)}{64M_3^2}+|x|^2(4-p_1^2)p_1^2\frac{U_1(t)}{48M_2M_4}+|x|^2(4-p_1^2)p_1\frac{U_1(t)}{24M_2M_4} \tag{2.6}$$

Since the class P is invariant under the rotation by lemma 1.5, one may assume without loss of generality that $p_1 = p \in [0, 2]$. Therefore, for $\eta = |x| \leq 1$;

$$|a_2a_4-a_3^2| = p^4\left\{\frac{U_1(t)U_3(t)}{48M_2M_4}+\frac{U_1^2(t)U_2(t)}{32M_2M_4}+\frac{U_1^4(t)}{96M_2M_4}+\frac{U_1^2(t)U_2(t)}{32M_3^2}+\frac{U_2^2(t)}{64M_3^2}+\frac{U_1^4(t)}{64M_3^2}\right.$$
$$+\frac{U_1^2(t)p^3}{24M_2M_4}+\frac{U_1(t)p}{6M_2M_4}+p^2(4-p^2)\eta\left\{\frac{U_1(t)U_2(t)}{24M_2M_4}+\frac{b^2(4t^3-t)}{12M_2M_4}+\frac{U_1^3(t)}{32M_2M_4}+\frac{U_1^2(t)}{32M_3^2}+\frac{U_1(t)U_2(t)}{32M_3^2}\right.$$
$$+\eta(2p^2-p^4)\frac{U_1^2(t)}{32M_3^2}+\eta^2(4-p^2)^2\frac{U_1(t)}{64M_3^2}+\eta^2(4-p^2)p^2\frac{U_1(t)}{48M_2M_4}\eta^2(4-p^2)p\frac{U_1(t)}{24M_2M_4} = J(\eta,p) \tag{2.7}$$

To maximize the function $J(\eta, p)$ on the closed region $[0, 2]x[0, 1]$, we find the first partial derivative of $J(\eta, p)$ with respect to η.

$$J^{'}(\eta,p) = p^2(4-p^2)\left\{\frac{U_1(t)U_2(t)}{24M_2M_4}+\frac{U_1^3(t)}{32M_2M_4}+\frac{U_1^3(t)}{32M_3^2}+\frac{U_1^2(t)}{32M_3^2}+\frac{U_1(t)U_2(t)}{32M_3^2}\right\}$$
$$+(2p^2-p^4)\frac{U_1^2(t)}{32M_3^2}+\eta(4-p^2)^2\frac{U_1(t)}{32M_3^2}+\eta(4-p^2)p^2\frac{U_1(t)}{24M_2M_4}+\eta(4-p^2)p\frac{U_1^2(t)}{12M_2M_4} \tag{2.8}$$

For $0 < \eta < 1$, and for a fixed p with $0 < p < 2$, it is observed that $J^{'}(\eta, p) > 0$. Therefore $J(\eta, p)$ becomes an increasing function of η and hence it cannot have a maximum value at any point in the interior of the closed region $[0, 1]x[0, 2]$. Moreover, for a fixed $p \in [0, 2]$

$$\max_{0\leq\eta\leq1} J(\eta,p) = J(1,p) = Q(p)$$

i.e the maximum value of $J(\eta, p)$ occurs at $\eta = 1$ and on further simplification of equation (2.7), it yields

$$Q(p,t) = \frac{p^4}{16}\left\{\frac{U_1(t)U_3(t)}{3M_2M_4} + \frac{U_1^2(t)U_2(t)}{2M_2M_4} + \frac{U_1^2(t)U_2(t)}{2M_3^2} + \frac{U_1^4(t)}{6M_2M_4} + \frac{U_1^4(t)}{4M_3^2} - \frac{2U_1(t)U_2(t)}{3M_2M_4} - \frac{U_1(t)U_2(t)}{2M_3^2} - \frac{U_1^3(t)}{2M_2M_4} - \frac{U_1^3(t)}{2M_3^2} - \frac{3U_1^2(t)}{4M_2M_4} - \frac{U_1^2(t)}{3M_2M_4}\right.$$

$$\left. + \frac{p^2}{4}\left\{\frac{2U_1(t)U_3(t)}{3M_2M_4} + \frac{U_1(t)U_2(t)}{2M_3^2} + \frac{U_1^3(t)}{2M_2M_4} + \frac{U_1^3(t)}{2M_3^2} + \frac{U_1^2(t)}{4M_3^2} + \frac{U_1^2(t)}{3M_2M_4}\right\} `+ \frac{U_1^2(t)}{4M_3^2}\right.$$

$$Q(p,t) = \frac{p^4\chi_1}{16} + \frac{p^2\chi_2}{4} + \frac{U_1^2(t)}{4M_3^2} \tag{2.9}$$

where:

$$\chi_1 = \frac{U_1(t)U_3(t)}{3M_2M_4} + \frac{U_1^2(t)U_2(t)}{2M_2M_4} + \frac{U_1^2(t)U_2(t)}{2M_3^2} + \frac{U_1^4(t)}{6M_2M_4} + \frac{U_1^4(t)}{4M_3^2} - \frac{2U_1(t)U_2(t)}{3M_2M_4} - \frac{U_1(t)U_2(t)}{2M_3^2} - \frac{U_1^3(t)}{2M_2M_4} - \frac{U_1^3(t)}{2M_3^2} - \frac{3U_1^2(t)}{4M_2M_4} - \frac{U_1^2(t)}{3M_2M_4}$$

$$\chi_2 = \frac{2U_1(t)U_3(t)}{3M_2M_4} + \frac{U_1(t)U_2(t)}{2M_3^2} + \frac{U_1^3(t)}{2M_2M_4} + \frac{U_1^3(t)}{2M_3^2} + \frac{U_1^2(t)}{4M_3^2} + \frac{U_1^2(t)}{3M_2M_4}$$

Suppose that $Q(p,t)$ has a maximum value in an interior of $p \in [0,2]$, then by elementary calculus,

$$Q'(p,t) = \frac{p^3\chi_1}{4} + \frac{p\chi_2}{2} \tag{2.10}$$

Next is to examine the sign of the function $Q'(p,t)$ depending on different cases of the signs of χ_1 and χ_2 as follows:

Case 1: Let $\chi_1 \geq 0$ and $\chi_2 \geq 0$, then $Q'(p,t) \geq 0$, so $Q(p,t)$ is an increasing function. Therefore $max\{Q(p,t) : p \in (0,2)\} = Q(2^-,t)$

$$= \frac{U_1^2(t)}{4M_3^2} + \chi_1 + \chi_2 \tag{2.11}$$

That is $max\ max\{\varphi(\eta) : 0 \leq \eta \leq 1\} : 0 < p < 2 = Q(2^-,t)$.

Case 2: Let $\chi_1 \leq 0$ and $\chi_2 \leq 0$, then $Q'(p,t) \leq 0$, so $Q(p,t)$ is a decreasing function. Therefore $max\{Q(p,t) : p \in (0,2)\} = Q(0^+,t)$

$$= \frac{t^2}{M_3^2} \tag{2.12}$$

Case 3: Let $\chi_1 > 0$ and $\chi_2 < 0$, then $p_0 = \sqrt{\frac{-2\chi_2}{\chi_1}}$ is a critical point of the function $Q(p,t)$.. We assume that $p_0 \in (0,2)$, since $Q''(p,t) > 0, p_0$ is a local minimum point of the function $Q(p,t)$ which simply means that the function $Q(p,t)$ cannot have a maximum value at p_0.

Case 4: Let $\chi_1 < 0$ and $\chi_2 > 0$, then p_0 is a critical point of the function $Q(p,t)$. Suppose $p_0 \in (0,2)$ since $Q''(p,t) < 0, p_0$ is a local maximum point of the function $Q(p,t)$ and maximum value occurs at $p = p_0$. Therefore $max\{Q(p,t) : p \in (0,2)\} = Q(p_0,t)$ where

$$Q(p_0,t) = \frac{t^2}{M_3^2} - \frac{\chi_2^2}{2\chi_1} \tag{2.13}$$

And using Equation (1.5)

$$\chi_1 = \frac{56t^4}{3M_2M_4} + \frac{8t^4}{3M_3^2} + \frac{28t^3}{3M_2M_4} + \frac{8t^3}{3M_3^2} + \frac{10t^2}{3M_2M_4} - \frac{6t^2}{3M_3^2} - \frac{4t}{3M_2M_4} - \frac{t}{M_3^2}$$

$$\chi_2 = \frac{34t^4}{3M_2M_4} + \frac{4t^3}{M_2M_4} + \frac{8t^3}{M_3^2} + \frac{4t^2}{M_2M_4} + \frac{t^2}{M_3^2} + \frac{t}{M_3^2}$$

Hence, from Equations (2.11), (2.12) and (2.13), the proof of Theorem 2.1 is complete.

3. Conclusion

Conclusively, this study introduces a new subclass $S^*(\mu, \beta, \wp, U_n(t))$ of analytic function characterized by Opoola differential operator, utilizing the subordination principle as a foundational framework. It is worth noting that the newly defined subclass is a generalization of a subclass introduced by Dziok e tal in [10]. Also, the findings which is a new result in geometric function theory, provide upper bounds to the second Hankel determinant, establishing significant connection with Chebyshev polynomials and Toeplitz determinants.

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