A Comprehenshive study of General decay results of a swelling soils system with arbitrary local memory effects versus frictional damping

Abstract

It is well known that swelling of soils, drying of fibers etc. are problems related to porous media theory, and there have been several studies by introducing continuum theories for fluids infiltrating in elastic porous media. This paper is concerned with the mixed initial-boundary value problem for swelling porous-elastic system with complementary frictional dampings and memory effects. Here, the novelties is: the fundamental condition that g′(t) is controlled by g(t) is removed, while the condition is a necessity in the previous literature. Expansive soil will lead to serious engineering problems, such as uneven foundation settlement, which will lead to small or large cracks in the building structure. Therefore, it is important to study the methods to eliminate or at least minimize this damage.

Keywords: Swelling ; Viscoelastic damping; Frictional damping; Energy decay; Arbitrary.

MSC: 93D23; 35Q70; 35B40; 74F05.

1 Introduction

The model of swelling soils reads as

$$\begin{cases} \rho_z z_{tt} = P_{1x} - G_1 + F_1, \\ \rho_u u_{tt} = P_{2x} + G_2 + F_2, \end{cases} \tag{1.1}$$

which was first proposed by Ieşan [10] in 1991 and simplified by Quintanilla [16] in 2002, where the constituents z and u represent the displacement of the fluid and the elastic solid material respectively. The positive constant coefficients ρ_z and ρ_u are the densities of each constituent. The functions P_1, G_1, F_1 represent the partial tension, internal body forces, and eternal forces acting on the displacement respectively. Similar definition holds for P_2, G_2, F_2 but acting on the elastic solid.

In recent years, the decay results of solutions to (1.1) were studied extensively for different forms of F_i and G_i, $i = 1, 2$, see [1, 4, 5, 14, 17–19] and the references therein. Recently, Mustafa et al. [14] considered (1.1) with

$$\begin{aligned} &F_1 = G_1 = 0, \\ &F_2 = \int_0^t g(t-s)(a(x)u_x(s))_x ds, \quad G_2 = b(x)h(u_t(t)) \end{aligned}$$

in $[0, 1]$, i.e., the following system

$$\begin{cases} \rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} = 0, & (x,t) \in (0,1) \times (0,\infty), \\ \rho_u u_{tt} - a_3 u_{xx} - a_2 z_{xx} + \displaystyle\int_0^t g(t-s)(a(x)u_x(s))_x ds + b(x)h(u_t(t)) = 0, & (x,t) \in (0,1) \times (0,\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in [0,1], \\ z(x,0) = z_0(x), z_t(x,0) = z_1(x), & x \in [0,1], \\ u(0,t) = u(1,t) = z(0,t) = z(1,t) = 0. \end{cases} \tag{1.2}$$

The term $\int_0^t g(t-s)(a(x)u_x(s))_x ds + b(x)h(u_t(t))$ in the second equation is called a **local mixed-type damping**. As said in [14], this problem is of more interest because there is competition between the frictional damping, represented by the term $(h(u_t))$, and the viscoelastic damping represented by the integral term with the relaxation function g, we refer to [2, 6–8, 11–13, 15, 20] and the references therein for local mixed-type damping systems. In addition, some scholars [3, 9] have recently studied prove some new existence and uniqueness results of solutions for nonlinear fractional implicit integro-differential equations of Hadamard-Caputo type with fractional boundary conditions. The reasoning is inspired by diverse classical fixed point theory, such as the Schauder and Banach fixed point theorems. The theoretical findings are illustrated through an example.

The main purpose of this paper is to get the decay of the energy functional $E(t)$ by weaken the assumptions (A2)[1] in [14], i.e., we weaken g to a absolutely continuous function and drop the assumption (1.5), i.e., we make the following assumption:

(I2) $g(t) : [0,\infty) \to (0,\infty)$ is a non-increasing and locally absolutely continuous function with meas($\mathcal{G}_0$)=0, $g'(t) \leq 0$ and (1.4) holds, where

$$\mathcal{G}_0 := \{s \geq 0; g(s) > 0, g'(s) = 0\}.$$

Next we state the main results of this paper. The well-posedness of the system (1.2) can be found

[1]The assumptions (A1)-(A3) of [14] can be found as follows:

(A1) $a, b : [0,1] \to [0,\infty)$ are such that

$$a \in C^1([0,1]), \quad b \in L^\infty([0,1]), \quad a(0) > 0, \quad \inf_{x\in[0,1]}[a(x) + b(x)] = c_0 > 0. \tag{1.3}$$

(A2) $g : [0,\infty) \to (0,\infty)$ is a C^1 function satisfying

$$a_0 = a_3 - \frac{a_2^2}{a_1} - \ell > 0, \tag{1.4}$$

and

$$g'(t) \leq -\xi(t)H_1(g(t)), \quad \forall\, t \geq 0, \tag{1.5}$$

where $\ell = \|a\|_\infty \left(\int_0^\infty g(s)ds\right)$ and ξ is a positive non-increasing differentiable function.

(A3) $h : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying, for some $c_1, c_2 > 0$,

$$\begin{aligned} &s^2 + h^2(s) \leq H_2^{-1}(sh(s)) && \text{for all } |s| \leq r_2, \\ &c_1|s| \leq |h(s)| \leq c_2|s| && \text{for all } |s| \geq r_2. \end{aligned} \tag{1.6}$$

Here $H_i : (0,\infty) \to (0,\infty)$ $(i = 1,2)$ are C^1 functions which are linear or strictly increasing and strictly convex C^2 functions on $(0, r_i]$ with $H_i(0) = H_i'(0) = 0$.

in [14, Theorem 2.2], and the energy functional can be defined by

$$\begin{aligned} E(t) :=& \frac{1}{2}\int_0^1 \left[\rho_z z_t^2 + a_1 z_x^2 + \rho_u u_t^2 + \left(a_3 - a(x)\int_0^t g(s)ds\right)u_x^2 + 2a_2 z_x u_x\right]dx \\ &+ \frac{1}{2}(g\circ u_x)(t) \end{aligned} \tag{1.7}$$

where

$$(g\circ u_x)(t) := \int_0^1 a(x)\int_0^t g(t-s)|u_x(t)-u_x(s)|^2 dsdx.$$

Theorem 1.1. *Assume (A1), (I2) and (A3) Hold. Then, the solution energy $E(t)$ of the system* (1.2) *satisfies,*

$$E(t) \le CE(0)\widehat{H_2}^{-1}((t+1)^{-1}),\ t\ge 0, \tag{1.8}$$

where $\widehat{H_2}(s)\sim H_2'(s)s$($a\sim b$: there exists positive constants c_1 and c_2 such that $c_1 a\le b\le c_2 a$).

In [14, Example 4.3], the authors got the following results: Assume that (A2) and (A3) hold with $H_i(t) = t^{\beta_i}$, $1\le\beta_1<2$ and $\beta_2\ge 1$, then by the main results of the paper, i.e., [14, Theorem 4.2],

$$E(t) \le \begin{cases} \vartheta e^{-\vartheta_1\int_0^t \xi(s)dx}, & \gamma = 1; \\ \overline{\vartheta}\left(1+\int_0^t \xi(s)dx\right)^{\frac{-1}{\gamma-1}}, & 1<\gamma<2; \\ \vartheta_2\left(1+\int_0^t \xi(s)dx\right)^{\frac{-1}{\gamma(\gamma-1)}}, & \gamma\ge 2, \end{cases}$$

where $\gamma = \min\{\beta_1,\beta_2\}$, $\vartheta,\vartheta_1,\vartheta_2,\overline{\vartheta}$ are positive constants, and $\xi(\cdot)$ is the function of (1.5).

Remark 1.2. We make several remarks for the above conclusions:

1. We remark that the value β_2 cannot be 1. In fact, if $\beta_2 = 1$, i.e., $H_2(t) = H_2^{-1}(t) = t$, then it follows from the first inequality of (1.6) that

   $$s^2 + h^2(s) \le sh(s).$$

   This is absurd since $s^2 + h^2(s)\ge 2sh(s)$.

2. If $\beta > 1$, there indeed exists $h(s)$ such that the assumption (A3) holds. In fact, for any $\alpha\in\left[\frac{1}{2\beta_2-1}, 2\beta_2-1\right]$, we take

   $$h(s) = \begin{cases} |s|^\alpha & \text{for all } |s|\le r_2, \\ r_2^{\alpha-1}|s| & \text{for all } |s|\ge r_2 \end{cases}$$

with $r_2 = \min\left\{2^{\frac{\beta_2}{\alpha+1-2\beta_2}},\ 2^{\frac{\beta_2}{\alpha+1-2\alpha\beta_2}}\right\}$. Then (1.6) holds, i.e., (A3) is true.

3. If $H_2(t) = t^{\beta_2}$ for $\beta_2 > 1$, by the first inequality of (1.6) in (A3), we get

$$s^2 + h^2(s) \leq H_2^{-1}(sh(s)) = (sh(s))^{\frac{1}{\beta_2}} \to 0 \text{ as } s \to 0,$$

which implies $\lim_{s\to 0} h(s) = 0$.

The rest of this paper is taken to the proofs of the above results.

2 Proof of the main results

In this section, we will prove the main results of this paper. Throughout the proofs, C represents some positive constants which may change from line to line. Through similar calculations, changing H to H_2 in [14, Lemma 4.1], we obtain corresponding result, for some specific $\varepsilon_1 > 0$,

$$\int_0^\infty E(s)H_2^{'}\left(\varepsilon_1\frac{E(s)}{E(0)}\right)ds = \frac{E(0)}{\varepsilon_1}\int_0^\infty \varepsilon_1\frac{E(s)}{E(0)}H_2^{'}\left(\varepsilon_1\frac{E(s)}{E(0)}\right)ds < \infty. \tag{2.1}$$

To get the theorem1.1, we need firstly prove

$$H_2'(\varepsilon_1\frac{E(t)}{E(0)})\varepsilon_1\frac{E(s)}{E(0)} \leq \frac{2C}{t+1},\ t \geq 0. \tag{2.2}$$

By the definition of H_2, we have

$$\frac{t}{2}H_2'(\varepsilon_1\frac{E(t)}{E(0)})\varepsilon_1\frac{E(t)}{E(0)} \leq \int_{\frac{t}{2}}^t H_2'(\varepsilon_1\frac{E(s)}{E(0)})\varepsilon_1\frac{E(s)}{E(0)}ds.$$

Owing to (2.1), the above inequality becomes

$$H_2'(\varepsilon_1\frac{E(t)}{E(0)})\varepsilon_1\frac{E(t)}{E(0)} \leq \frac{2C}{t}. \tag{2.3}$$

If $t \geq 1$, then $t - \frac{1}{2} \geq \frac{t}{2}$. (2.3) becomes

$$H_2'(\varepsilon_1\frac{E(t)}{E(0)})\varepsilon_1\frac{E(t)}{E(0)} \leq \frac{2C}{t-\frac{1}{2}+\frac{1}{2}} \leq \frac{2C}{\frac{t}{2}+\frac{1}{2}} = \frac{4C}{t+1}. \tag{2.4}$$

If $0 < t < 1$,

$$H_2'(\varepsilon_1\frac{E(t)}{E(0)})\varepsilon_1\frac{E(t)}{E(0)} \leq C = \frac{C}{t+1}(t+1) \leq \frac{2C}{t+1}. \tag{2.5}$$

By (2.4) and (2.5), we get (2.2). Then, by the definition of $\widehat{H_2}(s)$,

$$\widehat{H_2}\left(\varepsilon_1 \frac{E(t)}{E(0)}\right) \leq \frac{C}{t+1}$$
$$E(t) \leq C\widehat{H_2}^{-1}((t+1)^{-1})$$

Then, Theorem1.1 is proved.

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