UNDER PEER REVIEW

Normalized solutions to N-Laplacian systems with exponential critical growth in $\mathbb{R}^N$

Abstract

In this paper, we study the normalized solutions of N-Laplacian systems with exponential critical growth as follows

$$\begin{cases} -\Delta_N u = \lambda |u|^{N-2}u + G_u(u,v) \qquad \text{in } \mathbb{R}^N, \\ -\Delta_N v = \mu |v|^{N-2}v + G_v(u,v) \qquad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^N \, \mathrm{d}x = a^N, \qquad \int_{\mathbb{R}^N} |v|^N \, \mathrm{d}x = b^N, \end{cases}$$

where $N \geq 2,\ a, b > 0,\ \lambda, \mu \in \mathbb{R}$ and the nonlinear terms G_u, G_v, the partial derivatives of the function G, have exponential critical growth in $\mathbb{R}^N$. By using the Schwarz symmetrization and the Trudinger-Moser inequality, we establish the existence of normalized solutions for the above system.

Keywords: N-Laplacian system; Exponential critical growth; the Trudinger-Moser inequality; Normalized solutions.

1 Introduction

In this paper, we are concerned with the existence of normalized solutions for the following N-Laplacian systems with exponential critical growth

$$\begin{cases} -\Delta_N u = \lambda |u|^{N-2}u + G_u(u,v) \qquad \text{in } \mathbb{R}^N, \\ -\Delta_N v = \mu |v|^{N-2}v + G_v(u,v) \qquad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^N \, \mathrm{d}x = a^N, \qquad \int_{\mathbb{R}^N} |v|^N \, \mathrm{d}x = b^N, \end{cases} \tag{1.1}$$

where $\Delta_N u = \mathrm{div}(|\nabla u|^{N-2}\nabla u)$ is the N-Laplacian operator, $N \geq 2$, $a, b > 0$, $\lambda, \mu \in \mathbb{R}$ and the nonlinear terms G_u, G_v, the partial derivatives of the function G, have exponential critical growth in $\mathbb{R}^N$. Here the definition of exponential critical growth is motivated by Trudinger-Moser inequality (see e.g. [24, 30]).

Therefore, we naturally have the notion of G_u and G_v, which have critical growth of exponential type, i.e., there exists $\alpha_0 > 0$ such that

$$\lim_{|w|\to+\infty} \frac{|G_u(w)|}{\exp(\alpha|w|^{\frac{N}{N-1}})} = \lim_{|w|\to+\infty} \frac{|G_v(w)|}{\exp(\alpha|w|^{\frac{N}{N-1}})} = \begin{cases} 0, & \text{for any } \alpha > \alpha_0, \\ +\infty, & \text{for any } \alpha < \alpha_0, \end{cases} \tag{1.2}$$

where $w = (u, v) \in \mathbb{R}^2$.

The N-Laplacian operator plays an important role in many areas of physics (see e.g. [21, 26, 32] and the references therein), notably in fluid dynamics, electromagnetism and astronomy, because it can be used to accurately describe the known and unknown phenomenons of fluid potentials, electric, and gravitational in physics. In particular, for the case $N = 2$, system (1.1) also comes from the study of stationary wave solution for the following nonlinear Schrödinger system

$$\begin{cases} i\partial_t \Psi_1 + \Delta\Psi_1 + G_1(\Psi_1, \Psi_2) = 0 & \text{in } (0, \infty) \times \mathbb{R}^N, \\ i\partial_t \Psi_2 + \Delta\Psi_2 + G_2(\Psi_1, \Psi_2) = 0 & \text{in } (0, \infty) \times \mathbb{R}^N, \end{cases} \tag{1.3}$$

where solutions (Ψ_1, Ψ_2) of (1.3) with preserved L^2-masses are independent of $t \in (0, \infty)$. The above system could be used to the research of nonlinear optics and Bose-Einstein condensates (see e.g. [15, 17, 29]). From the corresponding physics, we know the solitary wave solutions have the following form

$$\Psi_1(t, x) = e^{-i\lambda t}u(x) \quad \text{and} \quad \Psi_2(t, x) = e^{-i\mu t}v(x),$$

where $\lambda, \mu \in \mathbb{R}$ and $u, v : \mathbb{R}^N \to \mathbb{R}$. On the other hand, let

$$G_u\big(e^{-i\lambda t}u(x), e^{-i\mu t}v(x)\big) = e^{-i\lambda t}G_1(u, v) \quad \text{and} \quad G_v\big(e^{-i\lambda t}u(x), e^{-i\mu t}v(x)\big) = e^{-i\mu t}G_2(u, v),$$

then (u, v, λ, μ) leads to elliptic system (1.1).

To obtain the normalized solutions of (1.1), we give the following definitions. Let

$$S(a, b) := \left\{(u, v) \in W^{1,N}(\mathbb{R}^N) \times W^{1,N}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^N \,\mathrm{d}x = a^N, \int_{\mathbb{R}^N} |v|^N \,\mathrm{d}x = b^N\right\}. \tag{1.4}$$

The energy functional $J : W^{1,N}(\mathbb{R}^N) \times W^{1,N}(\mathbb{R}^N) \to \mathbb{R}$ corresponding to (1.1) is defined by

$$J(u, v) := \frac{1}{N} \int_{\mathbb{R}^N} \big(|\nabla u|^N + |\nabla v|^N\big) \,\mathrm{d}x - \int_{\mathbb{R}^N} G(u, v) \,\mathrm{d}x. \tag{1.5}$$

We aim to find the critical point (u, v) of J on $S(a, b)$, then $\lambda, \mu \in \mathbb{R}$ will exist as Lagrange multipliers such that (u, v, λ, μ) solves the system (1.1). In particular, the point corresponding to the infimum of energy $J(u, v)$ among all the solutions is a ground state solution of (1.1).

When $N = 2$, Alves et al. [3] studied the normalized solutions for a nonlinear Schrödinger equation with critical growth as follows

$$\begin{cases} -\Delta u = \lambda u + f(u) & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \,\mathrm{d}x = a^2. \end{cases} \tag{1.6}$$

They pioneered the study of normalizing problems involving an exponential critical growth term with two-dimensional, which draws heavily on a version of the Trudinger-Moser inequality [5]. Moreover, Chang et al. [7] proved the existence of normalized ground state solutions of (1.6) for any $a > 0$ without the Ambrosetti-Rabinowitz condition. In addition, Chen and Tang [8] extended the result when $f(u) = \mu|u|^{p-2}u + (e^{u^2} - 1 - u^2)u$ and $p > 2$.

The study of systems with exponential critical nonlinearities also has received special attention in recent years (see e.g. [1, 2, 6, 12, 13]) for $N = 2$. Associated with the prescribed L^2-masses, Guo and Jeanjean [16] established the existence of two positive normalized solutions of (1.1), where $G_u(u,v) = \gamma_1|u|^{p-2}u + \beta r_1|u|^{r_1-2}u|v|^{r_2}$ and $G_v(u,v) = \gamma_2|v|^{q-2}v + \beta r_2|u|^{r_1}|v|^{r_2-2}v$. Moreover, Deng and Yu [9] treated the Schrödinger systems containing exponential nonlinearities, namely system (1.1) for $N = 2$ as follows

$$\begin{cases} -\Delta u + \lambda u = G_u(u,v) \qquad \text{in } \mathbb{R}^2, \\ -\Delta v + \mu v = G_v(u,v) \qquad \text{in } \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} |u|^2 \, \mathrm{d}x = a^2, \qquad \int_{\mathbb{R}^2} |v|^2 \, \mathrm{d}x = b^2. \end{cases} \tag{1.7}$$

When $N \geq 2$, we would mention that the N-Laplacian equations involving exponential critical growth have been widely investigated (see e.g. [10,11,21–23,27,32,33] and the references therein). Dou et al. [14] showed the existence of normalized mountain pass type solution for following N-Laplacian equations

$$\begin{cases} -\Delta_N u + \lambda|u|^{N-2}u = f(u) \qquad \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u|^N \, \mathrm{d}x = a^N, \end{cases} \tag{1.8}$$

where $N \geq 2$ and $f(u)$ is an exponential critical growth nonlinearity. To estimate the upper energy bound, they use the following condition on f

(f') There exists $\beta_0 > 0$, such that $\liminf\limits_{s\to\infty} sf(s)\exp(-\alpha_0|s|^{\frac{N}{N-1}}) \geq \beta_0$.

In this paper, we attempt to deal with normalized solutions of the exponential system (1.1) and assume that G_u, G_v are partial derivatives of a Carathéodry function G and the nonlinearities G_u, G_v satisfy:

(G_0) $G(w) \in C^2(\mathbb{R}^2, \mathbb{R})$, where $w = (u,v)$.

(G_1) $\lim\limits_{|w|\to 0} \dfrac{|G_u(w)|}{|w|^{2N-1}} = \lim\limits_{|w|\to 0} \dfrac{|G_v(w)|}{|w|^{2N-1}} = 0$ and $G(u,0) = G(0,v) = G_u(u,0) = G_v(0,v) = 0$ for all $u, v \in \mathbb{R}$.

(G_2) There exists $\alpha_0 > 0$ such that

$$\lim_{|w|\to+\infty} \frac{|G_u(w)|}{\exp(\alpha|w|^{\frac{N}{N-1}})} = \lim_{|w|\to+\infty} \frac{|G_v(w)|}{\exp(\alpha|w|^{\frac{N}{N-1}})} = \begin{cases} 0, & \text{for any } \alpha > \alpha_0, \\ +\infty, & \text{for any } \alpha < \alpha_0. \end{cases}$$

(G_3) There exist constants $p > 2N$ and $\sigma > 0$ such that

$$G(w) \geq \frac{\sigma}{p}|w|^p \qquad \text{for all } w \in \mathbb{R}^2.$$

(G_4) There exists a positive constant $\theta > 2N$ such that

$$0 < \theta G(w) \leq \nabla G(w) \cdot w \qquad \text{for } u, v \neq 0,$$

where $\nabla G(w) = \big(G_u(w), G_v(w)\big)$.

(G_5) $G_u(w)u > 0$ and $G_v(w)v > 0$ for $u, v \neq 0$.

(G_6) Let $\widetilde{G}(w) = \nabla G(w) \cdot w - NG(w)$ for all $w \in \mathbb{R}^2$. Then, $\nabla \widetilde{G}(w)$ exists and

$$\nabla \widetilde{G}(w) \cdot w \geq 2N\widetilde{G}(w) \qquad \text{for all } w \in \mathbb{R}^2.$$

To facilitate the corresponding conclusion for $w = (u, v)$, we introduce the following definitions: Let $E = W^{1,N}(\mathbb{R}^N) \times W^{1,N}(\mathbb{R}^N)$, and the corresponding norm is defined by

$$\|w\|_E^N := \int_{\mathbb{R}^N} \left(|\nabla u|^N + |u|^N\right) \mathrm{d}x + \int_{\mathbb{R}^N} \left(|\nabla v|^N + |v|^N\right) \mathrm{d}x,$$

where $w = (u, v) \in E$. Moreover,

$$|w|_N^N := \int_{\mathbb{R}^N} \left(|u|^N + |v|^N\right) \mathrm{d}x, \qquad |\nabla w|_N^N := \int_{\mathbb{R}^N} \left(|\nabla u|^N + |\nabla v|^N\right) \mathrm{d}x.$$

From (G_1) and (G_2), fix $q > N$, for any $\varepsilon > 0$ and $\alpha > \alpha_0$, there exists a constant $K_\varepsilon > 0$, which depends on $q, \alpha, \tau, \varepsilon$ and σ, such that

$$\big|\nabla G(w)\big| \leq \big|G_u(w)\big| + \big|G_v(w)\big| \leq \varepsilon |w|^{2N-1} + K_\varepsilon |w|^{q-1} \Big[\exp\Big(\alpha |w|^{\frac{N}{N-1}}\Big) - S_{N-2}(\alpha, w)\Big] \quad \text{for all } w \in \mathbb{R}^2, \tag{1.9}$$

and by (G_4), we have

$$\big|G(w)\big| \leq \varepsilon |w|^{2N} + K_\varepsilon |w|^{q} \Big[\exp\Big(\alpha |w|^{\frac{N}{N-1}}\Big) - S_{N-2}(\alpha, w)\Big] \qquad \text{for all } w \in \mathbb{R}^2, \tag{1.10}$$

where $S_{N-2}(\alpha, w) = \displaystyle\sum_{k=0}^{N-2} \frac{\alpha^k}{k!} |w|^{\frac{N}{N-1}k}$, $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$ and $\omega_{N-1}^{\frac{1}{N-1}}$ is the measure of the unit sphere in $\mathbb{R}^N$.

Since the lack of compactness of the Sobolev embedding in the whole $\mathbb{R}^N$, we can work in the space $W_r^{1,N}(\mathbb{R}^N)$, the subspace of $W^{1,N}(\mathbb{R}^N)$ formed by radially symmetric functions. The embedding $W_r^{1,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is compact for all $q \in (N, +\infty)$. Moreover, We denote

$$E_r := W_r^{1,N}(\mathbb{R}^N) \times W_r^{1,N}(\mathbb{R}^N) \quad \text{and} \quad S_r(a, b) := S(a, b) \cap E_r.$$

To use the Pohozaev mainfold, we define

$$P(u, v) := \int_{\mathbb{R}^N} \left(|\nabla u|^N + |\nabla v|^N\right) \mathrm{d}x - \int_{\mathbb{R}^N} \widetilde{G}(u, v)\, \mathrm{d}x, \tag{1.11}$$

where $\widetilde{G}(u, v) = \nabla G(u, v) \cdot (u, v) - NG(u, v)$. We denote

$$m(a, b) := \inf_{(u,v) \in \mathcal{P}(a,b)} J(u, v), \tag{1.12}$$

where

$$\mathcal{P}(a, b) := \Big\{(u, v) \in S(a, b) : P(u, v) = 0\Big\}. \tag{1.13}$$

Now, we introduce the action of the group $\mathbb{R}$ on $W^{1,N}(\mathbb{R}^N)$ defined by $s\star u(x) := e^s u(e^s x)$. Moreover, we denote $s\star(u, v)(x) := (s\star u, s\star v)$. For $w = (u, v) \in S(a, b)$ and $s \in \mathbb{R}$, the functional $\mathcal{F} : \mathbb{R} \times S(a, b) \to S(a, b)$ is defined by

$$\mathcal{F}(s, w)(x) := \big(e^s u(e^s x), e^s v(e^s x)\big).$$

It is easy to see that if $w \in S(a,b)$, then $\mathcal{F}(s,w) \in S(a,b)$ for all $s \in \mathbb{R}$.

Then, we consider the augmented functional $\widetilde{J} : \mathbb{R} \times S(a,b) \to \mathbb{R}$ defined by

$$\widetilde{J}(s,w) := J\big(\mathcal{F}(s,w)\big) = \frac{e^{Ns}}{N} \int_{\mathbb{R}^N} \left(|\nabla u|^N + |\nabla v|^N\right) \mathrm{d}x - \frac{1}{e^{Ns}} \int_{\mathbb{R}^N} G\big(e^s u(x), e^s v(x)\big) \mathrm{d}x, \tag{1.14}$$

from where it follows that

$$\partial_s \widetilde{J}(s,w) = e^{Ns}|\nabla w|_N^N - \int_{\mathbb{R}^N} \widetilde{G}\big(\mathcal{F}(s,w)\big) \mathrm{d}x = P\big(\mathcal{F}(s,w)\big). \tag{1.15}$$

The Main result of this paper is as follows:

Theorem 1.1. *Suppose* (G_0)-(G_5) *are satisfied. Then there exists* $\sigma_1 = \sigma_1(a,b) > 0$ *such that system (1.1) possesses a weak solution* (u,v,λ,μ) *with* $\lambda, \mu < 0$ *and* $u, v \in W^{1,N}(\mathbb{R}^N)$ *for any* $\sigma \geq \sigma_1$.

Moreover, if (G_6) *is also satisfied, then (u,v) is a nontrivial ground state solution of system (1.1).*

The paper is organized as follows: In Section 2, we give some preliminaries, such as version of Trudinger-Moser inequality and the properties about exponential critical nonlinearities. In Section 3, we verify the mountain-pass geometrical structure to obtain the existence of Palais-Smale-Pohozaev sequence and estimate the minimax level. Finally, we proof Theorem 1.1 in Section 4.

Notation:

- $|\ |_p$ denotes the norm of the Lebesgue space $L^p(\mathbb{R}^N)$, where $|u|_p := \big(\int_{\mathbb{R}^N} |u|^p \mathrm{d}x\big)^{\frac{1}{p}}$ for $1 \leq p < \infty$.
- $\|\ \|$ denotes the norm of the Sobolev space $W^{1,N}(\mathbb{R}^N)$, $\|u\| := \Big(\int_{\mathbb{R}^N} |\nabla u|^N \mathrm{d}x + \int_{\mathbb{R}^N} |u|^N \mathrm{d}x\Big)^{\frac{1}{N}}$.
- $o_n(1)$ denotes a real sequence with $o_n(1) \to 0$ as $n \to +\infty$.
- Denote $C, C_1, C_2, \cdots$ are universal positive constants.

2 Preliminaries

In this section, we introduce some preliminary results. First of all, let us recall the following version of Trudinger-Moser inequality in $\mathbb{R}^N$ as stated in [11].

Proposition 2.1. *([11, Lemma 1]) If* $N \geq 2$, $\alpha > 0$ *and* $u \in W^{1,N}(\mathbb{R}^N)$, *then*

$$\int_{\mathbb{R}^N} \left[\exp\left(\alpha |u|^{\frac{N}{N-1}}\right) - S_{N-2}(\alpha, u)\right] \mathrm{d}x < \infty.$$

Moreover, if $|\nabla u|_N^N \leq 1$, $|u|_N \leq M < \infty$ *and* $\alpha < \alpha_N$, *then there exists a constant* $C = C(N, M, \alpha)$, *which depends only on* N, M *and* α, *such that*

$$\int_{\mathbb{R}^N} \left[\exp\left(\alpha |u|^{\frac{N}{N-1}}\right) - S_{N-2}(\alpha, u)\right] \mathrm{d}x \leq C.$$

To study the N-Laplacian systems, by the definition of $w = (u,v)$, we immediately give the following inequalities.

Lemma 2.1. *If $N \geq 2$, $\alpha > 0$ and $w = (u, v) \in E$, then*

$$\int_{\mathbb{R}^N} \left[\exp\left(\alpha|w|^{\frac{N}{N-1}}\right) - S_{N-2}(\alpha, w)\right] \mathrm{d}x < \infty.$$

Moreover, if $|\nabla w|_N^N \leq 1$, $|u|_N \leq K_1 < \infty$, $|v|_N \leq K_2 < \infty$ and $\alpha < \alpha_N$, then there exists a constant $C = C(N, K_1, K_2, \alpha)$, which depends only on N, a, b and α, such that

$$\int_{\mathbb{R}^N} \left[\exp\left(\alpha|w|^{\frac{N}{N-1}}\right) - S_{N-2}(\alpha, w)\right] \mathrm{d}x \leq C.$$

Proof. We will need the following inequality: for any $q > 0$, there exists a positive constant $C(q)$, which depends only on q, such that

$$|a + b|^q \leq C(q)(|a|^q + |b|^q). \tag{2.1}$$

Hence, for $w = (u, v) \in E$ and $\alpha > 0$, using the above inequality and Proposition 2.1, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left[\exp\left(\alpha|w|^{\frac{N}{N-1}}\right) - S_{N-2}(\alpha, w)\right] \mathrm{d}x \\
=&\int_{\mathbb{R}^N} \sum_{k=N-1}^{\infty} \frac{\alpha^k}{k!} |w|^{\frac{N}{N-1}k} \mathrm{d}x \\
\leq&\int_{\mathbb{R}^N} \sum_{k=N-1}^{\infty} \frac{\alpha^k}{k!} C(N, k)\left(|u|^{\frac{N}{N-1}k} + |v|^{\frac{N}{N-1}k}\right) \mathrm{d}x \\
\leq&C_1 \int_{\mathbb{R}^N} \left[\exp\left(\alpha|u|^{\frac{N}{N-1}}\right) - S_{N-2}(\alpha, u)\right] \mathrm{d}x + C_1 \int_{\mathbb{R}^N} \left[\exp\left(\alpha|v|^{\frac{N}{N-1}}\right) - S_{N-2}(\alpha, v)\right] \mathrm{d}x \\
<&\infty.
\end{aligned}$$

Moreover, by $|\nabla u|_N^N + |\nabla v|_N^N = |\nabla w|_N^N \leq 1$ and Proposition 2.1, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left[\exp\left(\alpha|w|^{\frac{N}{N-1}}\right) - S_{N-2}(\alpha, w)\right] \mathrm{d}x \\
\leq&C_1 \int_{\mathbb{R}^N} \left[\exp\left(\alpha|u|^{\frac{N}{N-1}}\right) - S_{N-2}(\alpha, u)\right] \mathrm{d}x + C_1 \int_{\mathbb{R}^N} \left[\exp\left(\alpha|v|^{\frac{N}{N-1}}\right) - S_{N-2}(\alpha, v)\right] \mathrm{d}x \\
\leq&C,
\end{aligned}$$

where $C = C(N, a, b, \alpha) > 0$.

□

Lemma 2.2. *Assume that $\{w_n\} \subset E$ is bounded and*

$$\limsup_{n \to +\infty} |\nabla w_n|_N^N \in \left(0, \left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}\right).$$

Then for any $\alpha > \alpha_0$ close to α_0, there exists $t > 1$ close to 1 and $C > 0$ satisfying

$$\int_{\mathbb{R}^N} \left[\exp\left(\alpha|w_n|^{\frac{N}{N-1}}\right) - S_{N-2}(\alpha, w_n)\right]^t \mathrm{d}x \leq C \quad \text{for all } n \in \mathbb{N}.$$

Proof. Let

$$\beta := \limsup_{n\to+\infty} |\nabla w_n|_N^{\frac{N}{N-1}}.$$

We know that $\beta \in \big(0, \frac{\alpha_N}{\alpha_0}\big)$. It is easy to find some $\eta > 0$ such that $\beta < \frac{\alpha_N}{\alpha_0+\eta}$. Since $\alpha > \alpha_0$ close to α_0 and $t > 1$ close to 1, we write $\alpha = \alpha_0 + \varepsilon$ and $t \in (1, 1+\varepsilon)$. Then, we obtain

$$\limsup_{n\to+\infty} t\alpha|\nabla w_n|_N^{\frac{N}{N-1}} \le \limsup_{n\to+\infty} (1+\varepsilon)(\alpha_0+\varepsilon)|\nabla w_n|_N^{\frac{N}{N-1}} \le (\alpha_0+\eta)\beta < \alpha_N, \tag{2.2}$$

where $\varepsilon \in \Big(0, \min\{\frac{\eta}{\alpha_0+2}, 1\}\Big)$.

Therefore, by (2.2) and Lemma 2.1, there exists $C = C(t, m, N, a, b) > 0$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Big[\exp\Big(\alpha|w_n|^{\frac{N}{N-1}}\Big) - S_{N-2}(\alpha, w_n)\Big]^t \,\mathrm{d}x \\
\le &\int_{\mathbb{R}^N} \Big[\exp\Big(t\alpha|w_n|^{\frac{N}{N-1}}\Big) - S_{N-2}(t\alpha, w_n)\Big] \,\mathrm{d}x \\
= &\int_{\mathbb{R}^N} \Big\{\exp\Big[t\alpha|\nabla w_n|_N^{\frac{N}{N-1}}\Big(\frac{|w_n|}{|\nabla w_n|_N}\Big)^{\frac{N}{N-1}}\Big] - S_{N-2}\Big(t\alpha|\nabla w_n|_N^{\frac{N}{N-1}}, \frac{|w_n|}{|\nabla w_n|_N}\Big)\Big\} \,\mathrm{d}x \\
\le &\, C \qquad \text{for all } n \in \mathbb{N}.
\end{aligned}$$

The proof is complete. □

Corollary 2.1. *Assume that $w_n \rightharpoonup w$ in E and*

$$\limsup_{n\to+\infty} |\nabla(w_n - w)|_N^N \in \Big(0, \big(\frac{\alpha_N}{\alpha_0}\big)^{N-1}\Big).$$

Then for any $\alpha > \alpha_0$ close to α_0, there exists $t > 1$ close to 1 and $C > 0$ satisfying

$$\int_{\mathbb{R}^N} \Big[\exp\Big(\alpha|w_n|^{\frac{N}{N-1}}\Big) - S_{N-2}(\alpha, w_n)\Big]^t \,\mathrm{d}x \le C \qquad \text{for all } n \in \mathbb{N}.$$

Proof. Let $z_n = w_n - w$. By Lemma 2.2, we know that the sequence $\{\exp(\alpha|z_n|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, z_n)\}$ is bounded in $L^t(\mathbb{R}^N)$ for some $\alpha > \alpha_0$ close to α_0 and $t > 1$ close to 1. Thus, combining with the inequality (2.1), there exists $C > 0$ verifying

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Big[\exp\Big(\alpha|w_n|^{\frac{N}{N-1}}\Big) - S_{N-2}(\alpha, w_n)\Big]^t \,\mathrm{d}x \\
\le &\int_{\mathbb{R}^N} \Big[\exp\Big(t\alpha|z_n + w|^{\frac{N}{N-1}}\Big) - S_{N-2}(t\alpha, z_n + w)\Big] \,\mathrm{d}x \\
= &\int_{\mathbb{R}^N} \sum_{k=N-1}^{\infty} \frac{t^k\alpha^k}{k!} |z_n + w|^{\frac{N}{N-1}k} \,\mathrm{d}x \\
\le &\int_{\mathbb{R}^N} \sum_{k=N-1}^{\infty} \frac{t^k\alpha^k}{k!} C(N,k)\Big(|z_n|^{\frac{N}{N-1}k} + |w|^{\frac{N}{N-1}k}\Big) \,\mathrm{d}x
\end{aligned}$$

$$\le C_1 \int_{\mathbb{R}^N} \left[\exp\left(t\alpha |z_n|^{\frac{N}{N-1}}\right) - S_{N-2}(t\alpha, z_n)\right] \mathrm{d}x + C_1 \int_{\mathbb{R}^N} \left[\exp\left(t\alpha |w|^{\frac{N}{N-1}}\right) - S_{N-2}(t\alpha, w)\right] \mathrm{d}x$$
$$\le C.$$

The proof is complete. □

Lemma 2.3. *Assume that* (G_0)*-*(G_2) *hold. Let* $\{w_n\}$ *be a sequence with* $w_n = (u_n, v_n) \in E_r$ *such that* $w_n \rightharpoonup w$ *in* E_r *and*

$$\limsup_{n\to+\infty} |\nabla(w_n - w)|_N^N \in \left(0, (\frac{\alpha_N}{\alpha_0})^{N-1}\right).$$

Then we have

$$G(w_n) \to G(w) \quad \text{and} \quad \nabla G(w_n)\cdot w_n \to \nabla G(w)\cdot w \quad \text{in } L^1(\mathbb{R}^N).$$

Moreover, if (G_5) *holds, we have*

$$G_u(w_n)u_n \to G_u(w)u \quad \text{and} \quad G_v(w_n)v_n \to G_v(w)v \quad \text{in } L^1(\mathbb{R}^N).$$

Proof. By Corollary 2.1, we know that the sequence $\{\exp(\alpha|w_n|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, w_n)\}$ is bounded in $L^t(\mathbb{R}^N)$ for some $\alpha > \alpha_0$ close to α_0 and $t > 1$ close to 1. For the sake of simplicity, we set

$$h_n = \exp(\alpha|w_n|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, w_n)$$

Since $w_n \rightharpoonup w$ in E_r, there exists $w_n(x) \to w(x)$ a.e. in $\mathbb{R}^N$. Therefore, for some subsequence of $\{h_n\}$, still denoted by itself, we use the result in [20, Lemma 4.8] to obtain

$$h_n \rightharpoonup h = \exp(\alpha|w|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, w) \ \text{ in } L^t(\mathbb{R}^N). \tag{2.3}$$

For $t' = \frac{t}{t-1}$ and $w = (u, v)$, the compact embedding $W_r^{1,N}(\mathbb{R}^N) \hookrightarrow L^{qt'}(\mathbb{R}^N)$ ensures that

$$u_n \to u \quad \text{and} \quad v_n \to v \quad \text{in } L^{qt'}(\mathbb{R}^N).$$

Hence, we also have $|u_n|^2 \to |u|^2$ and $|v_n|^2 \to |v|^2$ in $L^{\frac{qt'}{2}}(\mathbb{R}^N)$ to derive that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(|w_n|^2 - |w|^2\right)^{\frac{qt'}{2}} \mathrm{d}x \\
=&\int_{\mathbb{R}^N} \left[(|u_n|^2 - |u|^2) + (|v_n|^2 - |v|^2)\right]^{\frac{qt'}{2}} \mathrm{d}x \\
\le& C_1 \int_{\mathbb{R}^N} \left(|u_n|^2 - |u|^2\right)^{\frac{qt'}{2}} \mathrm{d}x + C_1 \int_{\mathbb{R}^N} \left(|v_n|^2 - |v|^2\right)^{\frac{qt'}{2}} \mathrm{d}x \quad \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

which implies that

$$|w_n|^2 \to |w|^2 \quad \text{in } L^{\frac{qt'}{2}}(\mathbb{R}^N).$$

And so,

$$|w_n|^q \to |w|^q \quad \text{in } L^{t'}(\mathbb{R}^N). \tag{2.4}$$

Thus, from (2.3), (2.4) and the Hölder inequality, we have

$$\int_{\mathbb{R}^N} \left| |w_n|^q h_n(x) - |w|^q h(x) \right| \mathrm{d}x$$

$$\leq \int_{\mathbb{R}^N} |w_n|^q \big|h_n(x) - h(x)\big| \, \mathrm{d}x + \int_{\mathbb{R}^N} \Big||w_n|^q - |w|^q\Big| h(x) \, \mathrm{d}x$$
$$\leq \Big(\int_{\mathbb{R}^N} |w_n|^{qt'} \, \mathrm{d}x \Big)^{\frac{1}{t'}} \Big(\int_{\mathbb{R}^N} |h_n(x) - h(x)|^t \, \mathrm{d}x \Big)^{\frac{1}{t}} + \int_{\mathbb{R}^N} \Big||w_n|^q - |w|^q\Big| h(x) \, \mathrm{d}x \quad \to 0 \quad \text{as } n \to \infty.$$

That is,

$$|w_n|^q h_n \to |w|^q h \quad \text{in } L^1(\mathbb{R}^N). \tag{2.5}$$

By (1.9) and (1.10), we obtain

$$\big|G(w_n)\big| \leq \varepsilon |w_n|^{2N} + K_\varepsilon |w_n|^q \Big[\exp\Big(\alpha |w_n|^{\frac{N}{N-1}} \Big) - S_{N-2}(\alpha, w_n) \Big] \quad \text{for all } n \in \mathbb{N}, \tag{2.6}$$

and

$$\big|\nabla G(w_n) \cdot w_n\big| \leq \varepsilon |w_n|^{2N} + K_\varepsilon |w_n|^q \Big[\exp\Big(\alpha |w_n|^{\frac{N}{N-1}} \Big) - S_{N-2}(\alpha, w_n) \Big] \quad \text{for all } n \in \mathbb{N}, \tag{2.7}$$

where $q > N$ and $\alpha > \alpha_0$ close to α_0. Therefore, by (2.5), (2.6), (2.7) and the radial compact embedding, we can use a variant of the Lebesgue Dominated Convergence Theorem to conclude that

$$G(w_n) \to G(w) \quad \text{in } L^1(\mathbb{R}^N),$$

and

$$\nabla G(w_n) \cdot w_n \to \nabla G(w) \cdot w \quad \text{in } L^1(\mathbb{R}^N).$$

Moreover, by (G_4) and (G_5), we know that

$$0 < G_u(w_n) u_n \leq \nabla G(w_n) \cdot w_n \quad \text{and} \quad 0 < G_v(w_n) v_n \leq \nabla G(w_n) \cdot w_n. \tag{2.8}$$

A similar argument works to show that

$$G_u(w_n) u_n \to G_u(w) u \quad \text{and} \quad G_v(w_n) v_n \to G_v(w) v \quad \text{in } L^1(\mathbb{R}^N).$$

□

Lemma 2.4. *Assume that* (G_0)*-*(G_2) *hold. Let* $\{w_n\}$ *be a sequence with* $w_n = (u_n, v_n) \in E_r$ *such that* $w_n \rightharpoonup w$ *in* E_r *and*

$$\limsup_{n \to +\infty} \big|\nabla (w_n - w)\big|_N^N \in \Big(0, \big(\frac{\alpha_N}{\alpha_0}\big)^{N-1}\Big).$$

Then, for any $\phi, \psi \in C_0^\infty(\mathbb{R}^N)$*, we have*

$$G_u(w_n)\phi \to G_u(w)\phi \quad \textit{and} \quad G_v(w_n)\psi \to G_v(w)\psi \quad \textit{in } L^1(\mathbb{R}^N).$$

Proof. By (1.9), we know that

$$\big|G_u(w_n)\big| \leq \varepsilon |w_n|^{2N-1} + K_\varepsilon |w_n|^{q-1} \Big[\exp\Big(\alpha |w_n|^{\frac{N}{N-1}} \Big) - S_{N-2}(\alpha, w_n) \Big] \quad \text{for all } n \in \mathbb{N},$$

where $\alpha > \alpha_0$ close to α_0 and $q > N$. Hence, for any $\phi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\big|G_u(w_n)\phi\big| \leq \big|G_u(w_n)\big||\phi| \leq \varepsilon |w_n|^{2N-1}|\phi| + K_\varepsilon |w_n|^{q-1}|\phi| \Big[\exp\Big(\alpha |w_n|^{\frac{N}{N-1}} \Big) - S_{N-2}(\alpha, w_n) \Big] \text{ for all } n \in \mathbb{N}.$$

Denote $\Omega = supp\phi$. Similar to the arguments in the proof of Lemma 2.3, we obtain

$$\int_\Omega |w_n|^{2N-1}|\phi| \, \mathrm{d}x \to \int_\Omega |w|^{2N-1}|\phi| \, \mathrm{d}x,$$

and

$$\int_\Omega |w_n|^{q-1}|\phi|\Big[\exp\Big(\alpha|w_n|^{\frac{N}{N-1}}\Big) - S_{N-2}(\alpha, w_n)\Big] \, \mathrm{d}x \to \int_\Omega |w|^{q-1}|\phi|\Big[\exp\Big(\alpha|w|^{\frac{N}{N-1}}\Big) - S_{N-2}(\alpha, w)\Big] \, \mathrm{d}x,$$

where as $n \to +\infty$. Then, we can apply a variant of the Lebesgue Dominated Convergence Theorem to conclude that

$$G_u(w_n)\phi \to G_u(w)\phi \quad \text{in } L^1(\mathbb{R}^N), \qquad \text{for any } \phi \in C_0^\infty(\mathbb{R}^N).$$

A similar argument shows that

$$G_v(w_n)\psi \to G_v(w)\psi \quad \text{in } L^1(\mathbb{R}^N), \qquad \text{for any } \psi \in C_0^\infty(\mathbb{R}^N).$$

□

Lemma 2.5. *Assume that* (G_0)-(G_2) *hold. If* (u, v, λ, μ) *is a solution of (1.1), then* $w = (u, v) \in \mathcal{P}(a, b)$.

Proof. We recall that Pohozaev identity shows any solution u of the problem $-\Delta_p u = f(u)$ should satisfy the following identity:

$$(N - p)\int_{\mathbb{R}^N} |\nabla u|^p \, \mathrm{d}x = Np\int_{\mathbb{R}^N} F(u) \, \mathrm{d}x,$$

where $F(t) = \int_0^t f(s) \, \mathrm{d}s$. Then, as $p = N$, we have $\int_{\mathbb{R}^N} F(u) \, \mathrm{d}x = 0$.
Since (u, v, λ, μ) is a solution of (1.1), it follows that

$$\begin{cases} -\Delta_N u = \lambda|u|^{N-2}u + G_u(u, v) \\ -\Delta_N v = \mu|v|^{N-2}v + G_v(u, v) \end{cases} \tag{2.9}$$

On one hand, by (2.9), it yields that

$$\int_{\mathbb{R}^N} \left(|\nabla u|^N + |\nabla v|^N\right) \mathrm{d}x = \lambda\int_{\mathbb{R}^N} |u|^N \, \mathrm{d}x + \mu\int_{\mathbb{R}^N} |v|^N \, \mathrm{d}x + \int_{\mathbb{R}^N} \nabla G(u, v) \cdot (u, v) \, \mathrm{d}x. \tag{2.10}$$

On the other hand, as $p = N$, we obtain

$$\int_{\mathbb{R}^N} \Big[\frac{\lambda}{N}|\nabla u|^N + \frac{\mu}{N}|\nabla v|^N + G(u, v)\Big] \, \mathrm{d}x = 0. \tag{2.11}$$

Combining (2.10) and (2.11), we deduce that $P(u, v) = 0$. Hence $w = (u, v) \in \mathcal{P}(a, b)$.

□

3 Mountain pass approach

In this section, we shall verify the mountain-pass geometrical structure to the constrained functional $J|_{S(a,b)}$ and estimate the upper bound of the mountain pass level.

3.1 Mountain pass structure and value

Lemma 3.1. *Assume that* (G_0)*-*(G_3) *hold. Let* $w = (u, v) \in S(a, b)$ *be arbitrary but fixed. Then, we have:*

(*i*) $\big|\nabla \mathcal{F}(s, w)\big|_N \to 0$ *and* $J\big(\mathcal{F}(s, w)\big) \to 0$, *as* $s \to -\infty$;
(*ii*) $\big|\nabla \mathcal{F}(s, w)\big|_N \to +\infty$ *and* $J\big(\mathcal{F}(s, w)\big) \to -\infty$, *as* $s \to +\infty$.

Proof. Since

$$\int_{\mathbb{R}^N} \big|\nabla \mathcal{F}(s, w)(x)\big|^N \, \mathrm{d}x = e^{Ns} \int_{\mathbb{R}^N} |\nabla w|^N \, \mathrm{d}x,$$

we have

$$\big|\nabla \mathcal{F}(s, w)\big|_N \to 0 \quad \text{as } s \to -\infty \quad \text{and} \quad \big|\nabla \mathcal{F}(s, w)\big|_N \to +\infty \quad \text{as } s \to +\infty.$$

On the other hand, by a straightforward calculation, it follows that

$$\int_{\mathbb{R}^N} |s \star u(x)|^N \, \mathrm{d}x = a^N, \quad \int_{\mathbb{R}^N} |s \star v(x)|^N \, \mathrm{d}x = b^N,$$

and

$$\int_{\mathbb{R}^N} \big|\mathcal{F}(s, w)\big|^\xi \, \mathrm{d}x = e^{(\xi - N)s} \int_{\mathbb{R}^N} |w|^\xi \, \mathrm{d}x \qquad \text{for any } \xi \geq N. \tag{3.1}$$

Therefore, fixing $\xi > N$, we have

$$\big|\mathcal{F}(s, w)\big|_\xi \to 0 \quad \text{as } s \to -\infty.$$

By (1.10), we obtain

$$\big|G(w)\big| \leq \varepsilon |w|^{2N} + K_\varepsilon |w|^q \Big[\exp\Big(\alpha |w|^{\frac{N}{N-1}}\Big) - S_{N-2}(\alpha, w)\Big] \qquad \text{for all } w \in E, \tag{3.2}$$

where $\alpha > \alpha_0$ close to α_0 and $q > N$. Hence,

$$\big|G\big(\mathcal{F}(s, w)\big)\big| \leq \varepsilon \big|\mathcal{F}(s, w)\big|^{2N} + K_\varepsilon \big|\mathcal{F}(s, w)\big|^q \Big[\exp\Big(\alpha \big|\mathcal{F}(s, w)\big|^{\frac{N}{N-1}}\Big) - S_{N-2}\big(\alpha, \mathcal{F}(s, w)\big)\Big].$$

We choose $m > 0$ small enough such that $tm\alpha < \alpha_N$, and for s sufficiently negative to ensure

$$\Big|\nabla \Big(\frac{\mathcal{F}(s, w)}{m^{\frac{N-1}{N}}}\Big)\Big|_N^N \leq 1 \quad \text{and} \quad \Big|\frac{\mathcal{F}(s, w)}{m^{\frac{N-1}{N}}}\Big|_N = \Big(\frac{a^N + b^N}{m^{N-1}}\Big)^{\frac{1}{N}} < \infty.$$

Then, according to Lemma 2.1, there exists $C = C(t, m, w) > 0$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Big[\exp\Big(\alpha \big|\mathcal{F}(s, w)\big|^{\frac{N}{N-1}}\Big) - S_{N-2}\big(\alpha, \mathcal{F}(s, w)\big)\Big]^t \, \mathrm{d}x \\
&\leq \int_{\mathbb{R}^N} \Big[\exp\Big(t\alpha \big|\mathcal{F}(s, w)\big|^{\frac{N}{N-1}}\Big) - S_{N-2}\big(t\alpha, \mathcal{F}(s, w)\big)\Big] \, \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \Big[\exp\Big(tm\alpha \Big|\frac{\mathcal{F}(s, w)}{m^{\frac{N-1}{N}}}\Big|^{\frac{N}{N-1}}\Big) - S_{N-2}\Big(tm\alpha, \frac{\mathcal{F}(s, w)}{m^{\frac{N-1}{N}}}\Big)\Big] \, \mathrm{d}x \\
&\leq C,
\end{aligned}$$

and using the Hölder inequality, there exists a constant $D > 0$ such that

$$\int_{\mathbb{R}^N} \big|G\big(\mathcal{F}(s,w)\big)\big| \,\mathrm{d}x \le \varepsilon \int_{\mathbb{R}^N} \big|\mathcal{F}(s,w)\big|^{2N} \,\mathrm{d}x + D\Big(\int_{\mathbb{R}^N} \big|\mathcal{F}(s,w)\big|^{qt'} \,\mathrm{d}x\Big)^{\frac{1}{t'}},$$

where $t' = \frac{t}{t-1}$ and s is sufficiently negative. Now, by $qt' > N$ and (3.1), we derive that

$$\int_{\mathbb{R}^N} \big|G\big(\mathcal{F}(s,w)\big)\big| \,\mathrm{d}x \to 0 \quad \text{as } s \to -\infty,$$

from where it follows that

$$J\big(\mathcal{F}(s,w)\big) \to 0 \quad \text{as } s \to -\infty.$$

By (G_3), we know that there exist constants $\sigma > 0$ and $p > 2N$ such that

$$\begin{aligned} J\big(\mathcal{F}(s,w)\big) &\le \frac{1}{N}\big|\nabla\mathcal{F}(s,w)\big|_N^N - \sigma \int_{\mathbb{R}^N} \big|\mathcal{F}(s,w)\big|^p \,\mathrm{d}x \\ &= \frac{e^{Ns}}{N}\int_{\mathbb{R}^N} |\nabla w|^N \,\mathrm{d}x - \sigma e^{(p-N)s}\int_{\mathbb{R}^N} |w|^p \,\mathrm{d}x. \end{aligned}$$

Since $p > 2N$, the last inequality yields

$$J\big(\mathcal{F}(s,w)\big) \to -\infty \quad \text{as } s \to +\infty.$$

□

Lemma 3.2. *Assume that* (G_0)-(G_2) *and* (G_4) *hold. Then there exists* $K(a,b) > 0$ *small enough such that*

$$0 < \sup_{w\in A} J(w) < \inf_{w\in B} J(w)$$

with

$$A = \Big\{w \in S_r(a,b), \int_{\mathbb{R}^N} |\nabla w|^N \,\mathrm{d}x \le K(a,b)\Big\} \quad \textit{and} \quad B = \Big\{w \in S_r(a,b), \int_{\mathbb{R}^N} |\nabla w|^N \,\mathrm{d}x = 2K(a,b)\Big\}.$$

Proof. We recall the Gagliardo-Nirenberg inequality as stated in [25]: for any $s \ge N$, there exists a constant $C_{N,s} > 0$ such that

$$|u|_s \le C_{N,s}|\nabla u|_N^{\gamma_s}|u|_N^{1-\gamma_s} \qquad \text{for any } u \in W^{1,N}(\mathbb{R}^N), \tag{3.3}$$

where

$$\gamma_s := N\Big(\frac{1}{N} - \frac{1}{s}\Big).$$

For $w = (u,v) \in S_r(a,b)$, we know that there exist constants $C_1, C_2, C_3 > 0$ such that

$$\begin{aligned} |w|_s^s =& |u^2+v^2|_{\frac{s}{2}}^{\frac{s}{2}} \\ \le& C_1\big(|u|_N^s + |v|_N^s\big) \\ \le& C_2|\nabla u|_N^{s\gamma_s}|u|_N^{s(1-\gamma_s)} + C_3|\nabla v|_N^{s\gamma_s}|v|_N^{s(1-\gamma_s)} \\ =& C_2 a^{s(1-\gamma_s)}|\nabla u|_N^{s\gamma_s} + C_3 b^{s(1-\gamma_s)}|\nabla v|_N^{s\gamma_s}, \end{aligned}$$

hence, we obtain an estimate of $|w|_s$ for any $s \geq N$, there exists $C > 0$ as follows

$$|w|_s \leq C|\nabla w|_N^{\gamma_s} \qquad \text{where } \gamma_s = N(\frac{1}{N} - \frac{1}{s}). \tag{3.4}$$

Now, let $K(a,b) < \frac{1}{2}(\frac{\alpha_N}{\alpha_0})^{N-1}$, $\alpha > \alpha_0$ close to α_0, $w_1 = (u_1, v_1) \in A$ and $w_2 = (u_2, v_2) \in B$, we can find some $t > 1$ close to 1 such that $t\alpha|\nabla w_2|_N^{\frac{N}{N-1}} < \alpha_N$. Then, by Lemma 2.1 and the Hölder inequality, there exists $D_1 > 0$ such that for all $w_2 \in B$,

$$\int_{\mathbb{R}^N} |w_2|^q \Big[\exp\big(\alpha|w_2|^{\frac{N}{N-1}}\big) - S_{N-2}(\alpha, w_2)\Big] \,\mathrm{d}x \leq D_1|w_2|_{qt'}^q \,,$$

where $q > N$ and $t' = \frac{t}{t-1}$. Combining with the (G_0)-(G_2), for $\varepsilon > 0$ small enough such that $\varepsilon \leq \frac{1}{4NC^{2N}}$, we derive that

$$\int_{\mathbb{R}^N} G(w_2) \,\mathrm{d}x \leq \varepsilon|w_2|_{2N}^{2N} + D|w_2|_{qt'}^q,$$

where $D > 0$. Hence, by (3.4), we have

$$\int_{\mathbb{R}^N} G(w_2) \,\mathrm{d}x \leq \varepsilon C^{2N}|\nabla w_2|_N^N + DC^q|\nabla w_2|_N^{q-\frac{N}{t'}}. \tag{3.5}$$

Using the nonnegativity of $G(w_1)$ for all $w_1 \in E$ due to (G_4). For any $w_1 \in A$ and $w_2 \in B$, we have

$$\begin{aligned}
J(w_2) - J(w_1) =& \frac{1}{N}|\nabla w_2|_N^N - \frac{1}{N}|\nabla w_1|_N^N - \int_{\mathbb{R}^N} G(w_2) \,\mathrm{d}x + \int_{\mathbb{R}^N} G(w_1) \,\mathrm{d}x \\
\geq& \frac{1}{N}|\nabla w_2|_N^N - \frac{1}{N}|\nabla w_1|_N^N - \int_{\mathbb{R}^N} G(w_2) \,\mathrm{d}x \\
\geq& \frac{1}{N}K(a,b) - 2\varepsilon C^{2N}K(a,b) - 2^{\frac{qt'-N}{Nt'}} DC^q K(a,b)^{\frac{qt'-N}{Nt'}} \\
\geq& \frac{1}{2N}K(a,b) - 2^{\frac{qt'-N}{Nt'}} DC^q K(a,b)^{\frac{qt'-N}{Nt'}}.
\end{aligned}$$

Since $\frac{qt'-N}{Nt'} > 1$, fixing

$$K(a,b) < \min\left\{\frac{1}{2}(\frac{\alpha_N}{\alpha_0})^{N-1}, \Big(\frac{1}{2^{\frac{qt'-N+Nt'}{Nt'}} NDC^q}\Big)^{\frac{Nt'}{qt'-N-Nt'}}\right\}, \tag{3.6}$$

we get

$$\frac{1}{2N}K(a,b) - 2^{\frac{qt'-N}{Nt'}} DC^q K(a,b)^{\frac{qt'-N}{Nt'}} > 0,$$

which implies that

$$0 < \sup_{w\in A} J(w) < \inf_{w\in B} J(w).$$

□

Corollary 3.1. *Let $K(a,b) > 0$ be given in Lemma 3.2 and $w \in S_r(a,b)$. Then there holds that $J(w) > 0$ for $|\nabla w|_N^N \leq K(a,b)$. Moreover,*

$$J_* = \inf\left\{J(w) : w \in S_r(a,b) \quad \textit{and} \quad |\nabla w|_N^N = \frac{K(a,b)}{2}\right\} > 0.$$

Proof. Arguing as the last lemma, (3.5) is also valid after replacing w_2 by any $w_1 \in A$, then there holds that $J(w_1) > 0$ for all $w_1 \in A$.

For all $w \in S_r(a,b)$ with $|\nabla w|_N^N = \dfrac{K(a,b)}{2}$, we have

$$J(w) \geq \frac{1}{N}|\nabla w|_N^N - \varepsilon C^{2N}|\nabla w|_N^N - DC^q|\nabla w|_N^{q-\frac{N}{t'}} > 0,$$

by using the definition of $K(a,b) > 0$ given in (3.6).

□

Motivated by the ideas from Jeanjean [19], we introduce the mountain pass level as follows

$$c(a,b) := \inf_{h\in\Gamma}\max_{t\in[0,1]} J\big(h(t)\big), \tag{3.7}$$

where

$$\Gamma := \left\{h \in C\big([0,1], S_r(a,b)\big) : \big|\nabla h(0)\big|_N^N < \frac{K(a,b)}{2} \quad \text{and} \quad J\big(h(1)\big) < 0\right\}.$$

Then we have

Lemma 3.3. *Let $K(a,b) > 0$ be given in Lemma 3.2. Then the number $c(a,b)$ defined by (3.7) is positive, that is, $c(a,b) > 0$.*

Moreover, if (G_3) holds, we have $\lim_{\sigma\to+\infty} c(a,b) = 0$.

Proof. For any fixed $w_0 \in S_r(a,b)$, we can apply Lemma 3.1 to get two numbers $s_1 < 0$ and $s_2 > 0$ such that

$$\big|\nabla\mathcal{F}(s_1,w_0)\big|_N^N < \frac{K(a,b)}{2} \quad \text{and} \quad J\big(\mathcal{F}(s_2,w_0)\big) < 0. \tag{3.8}$$

We introduce the path $h_{w_0} : [0,1] \to S_r(a,b)$ defined by

$$h_{w_0}(t) = \mathcal{F}\big((1-t)s_1 + ts_2, w_0\big) \qquad \text{for any } t \in [0,1].$$

By (3.8), we know that $h_{w_0} \in \Gamma$ and hence $\Gamma \neq \emptyset$. From Corollary 3.1, we obtain $\big|\nabla h_{w_0}(1)\big|_N^N > K(a,b)$. Hence, there exists $t_0 \in (0,1)$ such that $\big|\nabla h_{w_0}(t_0)\big|_N^N = \frac{K(a,b)}{2}$, and so

$$\max_{t\in[0,1]} J\big(h(t)\big) \geq J\big(h_{w_0}(t_0)\big) \geq J_* > 0,$$

where $h \in \Gamma$ and J_* was given in Corollary 3.1. Therefore, $c(a,b) \geq J_* > 0$.

Moreover, by (G_3), we obtain

$$\begin{aligned} c(a,b) \leq \max_{t\in[0,1]} J\big(h_{w_0}(t)\big) &\leq \max_{s\in\mathbb{R}}\left\{\frac{e^{Ns}}{N}|\nabla w_0|_N^N - \sigma e^{(p-N)s}|w_0|_p^p\right\} \\ &= \max_{r\geq 0}\left\{\frac{r^N}{N}|\nabla w_0|_N^N - \sigma r^{p-N}|w_0|_p^p\right\}. \end{aligned}$$

Since $p > 2N$, there exists $C > 0$ such that

$$c(a,b) \leq C\Big(\frac{1}{\sigma}\Big)^{\frac{N}{p-2N}} \to 0 \qquad \text{as } \sigma \to +\infty.$$

The proof is complete.

□

Since the condition we considered in the present paper is of a mass super-critical case due to (G_1), which means that $G_u(w)$ and $G_v(w)$ would mass super-critical increasing near 0, a usual Palais-Smale sequence is not necessarily bounded in E_r. To overcome this problem, inspired by [19], we shall search for a Palais-Smale-Pohozaev sequence, which is a Palais-Smale sequence with an additional property $P(u,v) \to 0$, where $P(u,v)$ is defined by (1.11). Before proving the existence of a $(PSP)_{c(a,b)}$ sequence, we introduce the following definition in [31, Definition 2.21].

Definition 3.1. *Let X be a Banach space and $\Psi \in C^1(X,\mathbb{R})$, $d \in \mathbb{R}$. A sequence $\{u_n\} \subset X$ is called a $(PSP)_d$ sequence for Ψ if*

$$\Psi(u_n) \to d, \ \Psi'(u_n) \to 0, \ P(u_n) \to 0 \quad \text{as } n \to \infty.$$

The functional Ψ satisfies the $(PSP)_d$ condition if any $(PSP)_d$ sequence $\{u_n\}$ has a convergent subsequence.

In this part, we resort to the functional $\widetilde{J}(s,w)$ defined by (1.14) to get the desired sequence.

Arguing as in [19, Proposition 2.2], we know that there exists a (PS) sequence for $\widetilde{J}$ on $\mathbb{R} \times S_r(a,b)$ at the level $c(a,b)$, which is denoted by $\{(s_n, z_n)\}$, that is

$$J\big(\mathcal{F}(s_n,z_n)\big) = \widetilde{J}(s_n,z_n) \to c(a,b), \tag{3.9}$$

$$\partial_s \widetilde{J}(s_n,z_n) \to 0, \tag{3.10}$$

$$\|\partial_z \widetilde{J}(s_n,z_n)\|_{\left(T_{z_n}S_r(a,b)\right)^*} \to 0, \tag{3.11}$$

as $n \to +\infty$. Therefore, let $w_n = \mathcal{F}(s_n,z_n)$, we obtain that $\{w_n\} \subset S_r(a,b)$ is the (PS) sequence associated with the level $c(a,b)$ for J, where $w_n = (u_n, v_n)$. Thus, we have

$$J(w_n) \to c(a,b) \qquad \text{as } n \to +\infty, \tag{3.12}$$

and by the Lagrange multiplier rule, there exist some sequences $\{\lambda_n\}, \{\mu_n\} \subset \mathbb{R}$ such that

$$-\Delta_N u_n = \lambda_n |u_n|^{N-2} u_n + G_u(w_n) + o_n(1) \quad \text{and} \quad -\Delta_N v_n = \mu_n |v_n|^{N-2} v_n + G_v(w_n) + o_n(1). \tag{3.13}$$

Moreover, by (3.10) and (1.15), we have

$$P(w_n) = \int_{\mathbb{R}^N} |\nabla w_n|^N \,\mathrm{d}x + N \int_{\mathbb{R}^N} G(w_n)\,\mathrm{d}x - \int_{\mathbb{R}^N} \nabla G(w_n)\cdot w_n \,\mathrm{d}x \to 0 \qquad \text{as } n \to +\infty. \tag{3.14}$$

Hence, we naturally get the following result.

Lemma 3.4. *There exists a $(PSP)_{c(a,b)}$ sequence $\{w_n\} \subset S_r(a,b)$ with $w_n = (u_n, v_n)$, i.e.,*

$$J(w_n) \to c(a,b), \ J'(w_n) \to 0, \ P(w_n) \to 0 \quad \text{as } n \to +\infty.$$

3.2 $(PSP)_{c(a,b)}$ condition

Lemma 3.5. *Assume that* (G_0)-(G_4) *hold. Let* $\{w_n\} \subset S_r(a,b)$ *be the* $(PSP)_{c(a,b)}$ *sequence for* $J|_{S_r(a,b)}$, *then there holds*

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N} G(w_n)\,\mathrm{d}x \le \frac{N}{\theta-2N}\, c(a,b),$$

and

$$\limsup_{n\to+\infty} |\nabla w_n|_N^N \le \frac{N(\theta-N)}{\theta-2N}\, c(a,b).$$

Proof. By (3.12) and (3.14), we know that

$$\frac{1}{N}|\nabla w_n|_N^N - \int_{\mathbb{R}^N} G(w_n)\,\mathrm{d}x = c(a,b) + o_n(1), \tag{3.15}$$

and

$$|\nabla w_n|_N^N + N\int_{\mathbb{R}^N} G(w_n)\,\mathrm{d}x - \int_{\mathbb{R}^N} \nabla G(w_n)\cdot w_n\,\mathrm{d}x = o_n(1). \tag{3.16}$$

Combining (3.15), (3.16) and the condition (G_4), it follows that

$$Nc(a,b) + o_n(1) = \int_{\mathbb{R}^N} \nabla G(w_n)\cdot w_n\,\mathrm{d}x - 2N\int_{\mathbb{R}^N} G(w_n)\,\mathrm{d}x \ge (\theta-2N)\int_{\mathbb{R}^N} G(w_n)\,\mathrm{d}x.$$

Since $\theta > 2N$, we have

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N} G(w_n)\,\mathrm{d}x \le \frac{N}{\theta-2N}\, c(a,b). \tag{3.17}$$

Therefore, by (3.15) and (3.17), we obtain

$$\limsup_{n\to+\infty} |\nabla w_n|_N^N \le \frac{N(\theta-N)}{\theta-2N}\, c(a,b). \tag{3.18}$$

□

Remark 3.1. *Obviously, by (3.18) and Lemma 3.3, there exists* $\sigma_1 > 0$ *such that*

$$\limsup_{n\to+\infty} \big|\nabla(w_n - w)\big|_N^N \in \Big(0, \big(\frac{\alpha_N}{\alpha_0}\big)^{N-1}\Big) \qquad \text{for any } \sigma \ge \sigma_1. \tag{3.19}$$

Lemma 3.6. *Suppose* $\{w_n\} \subset S_r(a,b)$ *with* $w_n = (u_n, v_n)$ *is a* $(PSP)_{c(a,b)}$ *sequence for* $J|_{S_r(a,b)}$ *at level* $c(a,b) \neq 0$, *i.e.,*

$$J(w_n) \to c(a,b),\;\; J'(w_n) \to 0,\;\; P(w_n) \to 0 \quad \text{as } n \to +\infty.$$

Then up to a subsequence, $w_n \to w = (u,v)$ *in* E. *In particular,* $w \in S_r(a,b)$ *and there exist* $\lambda < 0$, $\mu < 0$ *such that* (u,v,λ,μ) *is a normalized mountain pass type solution of system (1.1).*

Proof. Lemma 3.5 yields that $|\nabla w_n|_N^N$ is bounded. By the definition of norm in E, we have

$$\|w_n\|_E^N = |\nabla w_n|_N^N + |u_n|_N^N + |v_n|_N^N,$$

where $w_n = (u_n, v_n) \in S_r(a,b)$. Hence, the $(PSP)_{c(a,b)}$ sequence $\{w_n\}$ is bounded in E. Up to a subsequence, we can assume that $w_n \rightharpoonup w := (u,v)$ in E_r. Since the embedding $W_r^{1,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is compact for all $q \in (N, +\infty)$, we also have $u_n \to u$ and $v_n \to v$ in $L^q(\mathbb{R}^N)$.

Next, we give the estimates of the Lagrange multipliers. Without loss of generality, we assume that $u, v \neq 0$. By Lemma 3.5 and (3.14), we obtain

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N} \nabla G(w_n)\cdot w_n \, \mathrm{d}x = \limsup_{n\to+\infty} \Big(\int_{\mathbb{R}^N} |\nabla w_n|^N \, \mathrm{d}x + N \int_{\mathbb{R}^N} G(w_n) \, \mathrm{d}x \Big) \leq \frac{N\theta}{\theta - 2N} \, c(a,b).$$

Then, by (2.8), we get

$$0 < \limsup_{n\to+\infty} \int_{\mathbb{R}^N} G_u(w_n) u_n \, \mathrm{d}x \leq \limsup_{n\to+\infty} \int_{\mathbb{R}^N} \nabla G(w_n)\cdot w_n \, \mathrm{d}x \leq \frac{N\theta}{\theta - 2N} \, c(a,b),$$
$$0 < \limsup_{n\to+\infty} \int_{\mathbb{R}^N} G_v(w_n) v_n \, \mathrm{d}x \leq \limsup_{n\to+\infty} \int_{\mathbb{R}^N} \nabla G(w_n)\cdot w_n \, \mathrm{d}x \leq \frac{N\theta}{\theta - 2N} \, c(a,b),$$

which imply that the sequence $\{\int_{\mathbb{R}^N} G_u(w_n)u_n \ \mathrm{d}x\}$ and $\{\int_{\mathbb{R}^N} G_v(w_n)v_n \ \mathrm{d}x\}$ are bounded. Since $J|'_{S_r(a,b)}(w_n) \to 0$, by the Lagrange multiplier rule, there exist some sequences $\{\lambda_n\}, \{\mu_n\} \subset \mathbb{R}$ such that

$$\begin{cases} \int_{\mathbb{R}^N} |\nabla u_n|^{N-2}\nabla u_n \cdot \nabla\phi - \lambda_n |u_n|^{N-2} u_n \phi - G_u(u_n, v_n)\phi \, \mathrm{d}x = o(1)\|\phi\|, \\ \int_{\mathbb{R}^N} |\nabla v_n|^{N-2}\nabla v_n \cdot \nabla\psi - \mu_n |v_n|^{N-2} v_n \psi - G_v(u_n, v_n)\psi \, \mathrm{d}x = o(1)\|\psi\|, \end{cases} \tag{3.20}$$

for any $\phi, \psi \in W_r^{1,N}(\mathbb{R}^N)$. Taking $\phi = u_n$ and $\psi = v_n$ with $w_n = (u_n, v_n) \in S_r(a,b)$, the numbers λ_n, μ_n must satisfy the equalities below

$$\lambda_n a^N = |\nabla u_n|_N^N - \int_{\mathbb{R}^N} G_u(w_n)u_n \, \mathrm{d}x + o_n(1) \quad \text{and} \quad \mu_n b^N = |\nabla v_n|_N^N - \int_{\mathbb{R}^N} G_v(w_n)v_n \, \mathrm{d}x + o_n(1). \tag{3.21}$$

Hence, the above arguments conclude that $\{\lambda_n\}, \{\mu_n\}$ are bounded sequences as well. Up to a subsequence, still denoted by $\{\lambda_n\}$ and $\{\mu_n\}$, we assume that $\lambda_n \to \lambda$, $\mu_n \to \mu$ in $\mathbb{R}$. Therefore, by the equalities (3.13), (3.20) and Lemma 2.4, we obtain

$$-\Delta_N u = \lambda |u|^{N-2}u + G_u(w) + o_n(1) \quad \text{and} \quad -\Delta_N v = \mu |v|^{N-2}v + G_v(w) + o_n(1). \tag{3.22}$$

In the following, we show that $w = (u, v) \neq (0, 0)$. Without loss of generality, we assume that $u = 0$ and claim that $v \neq 0$. Because otherwise, by Remark 3.1 and Lemma 2.3, we obtain

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} G(w_n) \, \mathrm{d}x = \int_{\mathbb{R}^N} G(w) \, \mathrm{d}x = 0, \tag{3.23}$$

and

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} \nabla G(w_n)\cdot w_n \, \mathrm{d}x = \int_{\mathbb{R}^N} \nabla G(w)\cdot w \, \mathrm{d}x = 0. \tag{3.24}$$

Thus, together with the limit $P(w_n) \to 0$, it yields $|\nabla w_n|_N^N \to 0$, from where it follows that

$$c(a,b) = \lim_{n\to+\infty} \Big[\frac{1}{N} \int_{\mathbb{R}^N} |\nabla w_n|^N \, \mathrm{d}x - \int_{\mathbb{R}^N} G(w_n) \, \mathrm{d}x \Big] = 0.$$

This contradicts $c(a,b) \neq 0$, which implies the claim $v \neq 0$. Moreover, by (G_1), we have $\nabla G(w)\cdot w = G_u(w)u + G_v(w)v = 0$. Similar to the above proof process, there exists a contradiction, implying that $u, v \neq 0$. Consequently, it is fully demonstrated that the weak limit w of $\{w_n\}$ is nontrivial. And thus

$$0 < |u|_N^N \leq \liminf_{n\to+\infty} |u_n|_N^N \quad \text{and} \quad 0 < |v|_N^N \leq \liminf_{n\to+\infty} |v_n|_N^N. \tag{3.25}$$

According to Lemma 2.3, we deduce that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} G(w_n)\ \mathrm{d}x=\int_{\mathbb{R}^N} G(w)\ \mathrm{d}x,\quad \lim_{n\to+\infty}\int_{\mathbb{R}^N} \nabla G(w_n)\cdot w_n\ \mathrm{d}x=\int_{\mathbb{R}^N} \nabla G(w)\cdot w\ \mathrm{d}x. \tag{3.26}$$

It follows from Lemma 2.5 that

$$P(u,v)=|\nabla w|_N^N+N\int_{\mathbb{R}^N} G(w)\ \mathrm{d}x-\int_{\mathbb{R}^N} \nabla G(w)\cdot w\ \mathrm{d}x=0. \tag{3.27}$$

Combining (3.26), (3.27) and the fact that $P(w_n)\to 0$, we have

$$\lim_{n\to+\infty}|\nabla w_n|_N^N=|\nabla w|_N^N. \tag{3.28}$$

Hence, $J(w)=\lim\limits_{n\to+\infty} J(w_n)=c(a,b)$. By [18, Lemma A.2], we know that $-\Delta_N u<0$ and $-\Delta_N v<0$. Thus, we have $\lambda<0$ and $\mu<0$. Then by (3.13), (3.22) and Lemma 2.3

$$\begin{aligned}\lim_{n\to+\infty}\left(|\nabla u_n|_N^N-\lambda_n|u_n|_N^N\right)&=\lim_{n\to+\infty}\int_{\mathbb{R}^N} G_u(u_n,v_n)u_n\ \mathrm{d}x\\ &=\int_{\mathbb{R}^N} G_u(u,v)u\ \mathrm{d}x\\ &=|\nabla u|_N^N-\lambda|u|_N^N,\end{aligned}$$

we obtain that $u_n\to u$ in $W^{1,N}(\mathbb{R}^N)$. Similarly, we can prove that $v_n\to v$ in $W^{1,N}(\mathbb{R}^N)$.

Now, we prove that the nontrivial solution w of system (1.1) satisfies $|u|_N^N=a^N$ and $|v|_N^N=b^N$. The limit $P(w_n)\to 0$ together with (3.21) leads to

$$\lambda_n|u_n|_N^N+\mu_n|v_n|_N^N=-N\int_{\mathbb{R}^N} G(w_n)\ \mathrm{d}x+o_n(1). \tag{3.29}$$

On the other hand, by (3.22) and $P(w)=0$, we have

$$\lambda a^N+\mu b^N=-N\int_{\mathbb{R}^N} G(w)\ \mathrm{d}x. \tag{3.30}$$

Combining (3.29), (3.30) and Lemma 2.3, we obtain $\lambda_n|u_n|_N^N+\mu_n|v_n|_N^N=\lambda a^N+\mu b^N$. By (3.25), we get the desired result.

□

4 Proof of Theorem 1.1

In this section, we shall complete the proof of Theorem 1.1. To prove the existence of a ground state solution to system (1.1), we need the following lemmas.

Lemma 4.1. *Assume that* (G_0) *and* (G_6) *hold. For any* $w\in S(a,b)$*, there exists a unique number* $s_w\in\mathbb{R}$ *such that*

$$\widetilde{J}(s_w,w)=\max_{s\in\mathbb{R}}\widetilde{J}(s,w)>0.$$

Moreover, s_w *is the unique number such that* $\mathcal{F}(s_w,w)\in\mathcal{P}(a,b)$.

Proof. First, by Lemma 3.1, we know that

$$\widetilde{J}(s,w) \to 0^+ \quad \text{as } s \to -\infty \quad \text{and} \quad \widetilde{J}(s,w) \to -\infty \quad \text{as } s \to +\infty.$$

Therefore, $\widetilde{J}(s,w)$ has a global maximum point $s_w \in \mathbb{R}$ such that $\widetilde{J}(s_w,w) = \max_{s\in\mathbb{R}} \widetilde{J}(s,w) > 0$, which means that $\partial_s \widetilde{J}(s_w,w) = 0$.

Next, we show the uniqueness of s_w. Obviously, by (1.15), we have

$$\begin{aligned}\partial_s \widetilde{J}(s,w) = P\big(\mathcal{F}(s,w)\big) =& \big|\nabla \mathcal{F}(s,w)\big|_N^N - \int_{\mathbb{R}^N} \widetilde{G}\big(\mathcal{F}(s,w)\big)\, \mathrm{d}x \\ =& e^{Ns}|\nabla w|_N^N - e^{Ns}\int_{\mathbb{R}^N} \frac{\widetilde{G}(e^s w)}{(e^s|w|)^{2N}}|w|^{2N}\, \mathrm{d}x =: L(s).\end{aligned}$$

By (G_6), $L(s)$ is strictly increasing in $\mathbb{R}$. Hence, s_w is the unique number such that $P\big(\mathcal{F}(s_w,w)\big) = 0$, from where it follows that $\mathcal{F}(s_w,w) \in \mathcal{P}(a,b)$.

□

Lemma 4.2. *Assume that* (G_0) *and* (G_6) *hold. Then, there holds* $c(a,b) = m(a,b)$.

Proof. Lemma 4.1 yields that $m(a,b) = \inf_{w\in\mathcal{P}(a,b)} \max_{s\in\mathbb{R}} J\big(\mathcal{F}(s,w)\big)$. Fixing $w \in S(a,b)$, for any $\mathcal{F}(s,w)$, there exists $h \in \Gamma$ such that

$$h(t) = \mathcal{F}\big((1-t)s_1 + ts_2, w\big) \quad t \in [0,1],$$

where two numbers $s_1 << -1$ and $s_2 >> 1$ satisfy $s = (1-t)s_1 + ts_2$. Then, we have

$$\max_{s\in\mathbb{R}} J\big(\mathcal{F}(s,w)\big) = \max_{t\in[0,1]} J\big(h(t)\big) \ge c(a,b).$$

This implies that $m(a,b) \ge c(a,b)$. On the other hand, fixing $w \in \mathcal{P}(a,b)$, for any $h \in \Gamma$, we have

$$\max_{t\in[0,1]} J\big(h(t)\big) \ge \inf_{w\in\mathcal{P}(a,b)} \max_{s\in\mathbb{R}} J\big(\mathcal{F}(s,w)\big),$$

which implies $c(a,b) \ge m(a,b)$. Hence, $m(a,b) = c(a,b)$.

□

Proof of Theorem 1.1: Under the assumptions (G_0)-(G_5), we prove that there exists a Palais-Smale-Pohozaev sequence for J constrained on $S_r(a,b)$ at level $c(a,b)$. Therefore, by Lemma 3.6, up to a subsequence, $(u_n, v_n) \to (u,v)$ in E. In particular, there exist $\lambda < 0$, $\mu < 0$ such that (u,v,λ,μ) is a normalized solution of system (1.1).

Moreover, we will prove that w is a ground state solution. Arguing as in [28] and using Lemma 4.2, it follows that

$$c(a,b) = m(a,b),$$

where $m(a,b)$ is defined by (1.12). Since $J(w) = c(a,b)$, the above analysis implies that w is a normalized ground state solution of system (1.1).

□

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