General decay results of a swelling soils system with arbitrary local memory effects versus frictional damping

Abstract

This paper is concerned with the mixed initial-boundary value problem for swelling porous-elastic system with complementary frictional dampings and memory effects. Here, one of the novelties is: the fundamental condition that $g'(t)$ is controlled by $g(t)$ is removed, while the condition is a necessity in the previous literature. The other distinct novelty is: for the exponential decay rates we only assume minimal conditions on $h(s)$.

Keywords: Swelling ; Viscoelastic damping; Frictional damping; Energy decay; Arbitrary.

MSC: 93D23; 35Q70; 35B40; 74F05.

1 Introduction

The model of swelling soils reads as

$$\begin{cases} \rho_z z_{tt} = P_{1x} - G_1 + F_1, \\ \rho_u u_{tt} = P_{2x} + G_2 + F_2, \end{cases} \tag{1.1}$$

which was first proposed by Ieşan [8] in 1991 and simplified by Quintanilla [14] in 2002, where the constituents z and u represent the displacement of the fluid and the elastic solid material respectively. The positive constant coefficients ρ_z and ρ_u are the densities of each constituent. The functions P_1, G_1, F_1 represent the partial tension, internal body forces, and eternal forces acting on the displacement respectively. Similar definition holds for P_2, G_2, F_2 but acting on the elastic solid.

In recent years, the decay results of solutions to (1.1) were studied extensively for different forms of F_i and G_i, $i = 1, 2$, see [1, 3, 4, 12, 15–17] and the references therein. Recently, Mustafa et al. [12] considered (1.1) with

$$\begin{aligned} &F_1 = G_1 = 0, \\ &F_2 = \int_0^t g(t-s)(a(x)u_x(s))_x ds, \quad G_2 = b(x)h(u_t(t)) \end{aligned}$$

in $[0, 1]$, i.e., the following system

$$\begin{cases} \rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} = 0, & (x,t) \in (0,1) \times (0,\infty), \\ \rho_u u_{tt} - a_3 u_{xx} - a_2 z_{xx} + \displaystyle\int_0^t g(t-s)(a(x)u_x(s))_x ds + b(x)h(u_t(t)) = 0, & (x,t) \in (0,1) \times (0,\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in [0,1], \\ z(x,0) = z_0(x), z_t(x,0) = z_1(x), & x \in [0,1], \\ u(0,t) = u(1,t) = z(0,t) = z(1,t) = 0. \end{cases} \tag{1.2}$$

The term $\int_0^t g(t-s)(a(x)u_x(s))_x ds + b(x)h(u_t(t))$ in the second equation is called a **local mixed-type damping**. As said in [12], this problem is of more interest because there is competition between the frictional damping, represented by the term $(h(u_t))$, and the viscoelastic damping represented by the integral term with the relaxation function g, we refer to [2, 5–7, 9–11, 13, 18] and the references therein for local mixed-type damping systems.

The main purpose of this paper is to get the decay of the energy functional $E(t)$ by weaken the assumptions (A2)[1] in [12], i.e., we weaken g to a absolutely continuous function and drop the assumption (1.5), i.e., we make the following assumption:

(I2) $g(t):[0,\infty)\to(0,\infty)$ is a non-increasing and locally absolutely continuous function with meas($\mathcal{G}_0$)=0, $g'(t)\le 0$ and (1.4) holds, where

$$\mathcal{G}_0 := \{s\ge 0; g(s)>0, g'(s)=0\}.$$

Next we state the main results of this paper. The well-posedness of the system (1.2) can be found in [12, Theorem 2.2], and the energy functional can be defined by

$$\begin{aligned} E(t) :=& \frac{1}{2}\int_0^1 \left[\rho_z z_t^2 + a_1 z_x^2 + \rho_u u_t^2 + \left(a_3 - a(x)\int_0^t g(s)ds\right)u_x^2 + 2a_2 z_x u_x\right]dx \\ &+ \frac{1}{2}(g\circ u_x)(t) \end{aligned} \tag{1.7}$$

where

$$(g\circ u_x)(t) := \int_0^1 a(x)\int_0^t g(t-s)|u_x(t)-u_x(s)|^2 ds dx.$$

[1]The assumptions (A1)-(A3) of [12] can be found as follows:

(A1) $a,b:[0,1]\to[0,\infty)$ are such that

$$a\in C^1([0,1]),\quad b\in L^\infty([0,1]),\quad a(0)>0,\quad \inf_{x\in[0,1]}[a(x)+b(x)] = c_0>0. \tag{1.3}$$

(A2) $g:[0,\infty)\to(0,\infty)$ is a C^1 function satisfying

$$a_0 = a_3 - \frac{a_2^2}{a_1} - \ell > 0, \tag{1.4}$$

and

$$g'(t)\le -\xi(t)H_1(g(t)),\quad \forall\, t\ge 0, \tag{1.5}$$

where $\ell = \|a\|_\infty\left(\int_0^\infty g(s)ds\right)$ and ξ is a positive non-increasing differentiable function.

(A3) $h:\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying, for some $c_1,c_2>0$,

$$\begin{aligned} &s^2+h^2(s)\le H_2^{-1}(sh(s)) && \text{for all } |s|\le r_2,\\ &c_1|s|\le|h(s)|\le c_2|s| && \text{for all } |s|\ge r_2. \end{aligned} \tag{1.6}$$

Here $H_i:(0,\infty)\to(0,\infty)$ $(i=1,2)$ are C^1 functions which are linear or strictly increasing and strictly convex C^2 functions on $(0,r_i]$ with $H_i(0)=H_i'(0)=0$.

Theorem 1.1. *Assume (A1), (I2) and (A3) Hold. Then, the solution energy $E(t)$ of the system* (1.2) *satisfies,*

$$E(t) \le CE(0)\widehat{H_2}^{-1}((t+1)^{-1}),\ t \ge 0, \tag{1.8}$$

where $\widehat{H_2}(s) \sim H_2'(s)s$($a \sim b$: there exists positive constants c_1 and c_2 such that $c_1 a \le b \le c_2 a$).

In [12, Example 4.3], the authors got the following results: Assume that (A2) and (A3) hold with $H_i(t) = t^{\beta_i}$, $1 \le \beta_1 < 2$ and $\beta_2 \ge 1$, then by the main results of the paper, i.e., [12, Theorem 4.2],

$$E(t) \le \begin{cases} \vartheta e^{-\vartheta_1 \int_0^t \xi(s)dx}, & \gamma = 1; \\ \overline{\vartheta}\left(1 + \int_0^t \xi(s)dx\right)^{\frac{-1}{\gamma-1}}, & 1 < \gamma < 2; \\ \vartheta_2\left(1 + \int_0^t \xi(s)dx\right)^{\frac{-1}{\gamma(\gamma-1)}}, & \gamma \ge 2, \end{cases}$$

where $\gamma = \min\{\beta_1, \beta_2\}$, $\vartheta, \vartheta_1, \vartheta_2, \overline{\vartheta}$ are positive constants, and $\xi(\cdot)$ is the function of (1.5).

Remark 1.2. We make several remarks for the above conclusions:

1. We remark that the value β_2 cannot be 1. In fact, if $\beta_2 = 1$, i.e., $H_2(t) = H_2^{-1}(t) = t$, then it follows from the first inequality of (1.6) that

$$s^2 + h^2(s) \le sh(s).$$

This is absurd since $s^2 + h^2(s) \ge 2sh(s)$.

2. If $\beta > 1$, there indeed exists $h(s)$ such that the assumption (A3) holds. In fact, for any $\alpha \in \left[\frac{1}{2\beta_2-1}, 2\beta_2 - 1\right]$, we take

$$h(s) = \begin{cases} |s|^\alpha & \text{for all } |s| \le r_2, \\ r_2^{\alpha-1}|s| & \text{for all } |s| \ge r_2 \end{cases}$$

with $r_2 = \min\left\{2^{\frac{\beta_2}{\alpha+1-2\beta_2}},\ 2^{\frac{\beta_2}{\alpha+1-2\alpha\beta_2}}\right\}$. Then (1.6) holds, i.e., (A3) is true.

3. If $H_2(t) = t^{\beta_2}$ for $\beta_2 > 1$, by the first inequality of (1.6) in (A3), we get

$$s^2 + h^2(s) \le H_2^{-1}(sh(s)) = (sh(s))^{\frac{1}{\beta_2}} \to 0 \text{ as } s \to 0,$$

which implies $\lim_{s\to 0} h(s) = 0$. Then it follows from Corollary **??** the energy functional $E(t)$ decay exponentially, which obviously improves the results of [12, Example 4.3].

The rest of this paper is taken to the proofs of the above results.

2 Proof of the main results

In this section, we will prove the main results of this paper. Throughout the proofs, C represents some positive constants which may change from line to line. Through similar calculations, changing H to H_2 in [12, Lemma 4.1], we obtain corresponding result, for some specific $\varepsilon_1 > 0$,

$$\int_0^\infty E(s)H_2'\left(\varepsilon_1\frac{E(s)}{E(0)}\right)ds = \frac{E(0)}{\varepsilon_1}\int_0^\infty \varepsilon_1\frac{E(s)}{E(0)}H_2'\left(\varepsilon_1\frac{E(s)}{E(0)}\right)ds < \infty. \tag{2.1}$$

To get the theorem1.1, we need firstly prove

$$H_2'(\varepsilon_1\frac{E(t)}{E(0)})\varepsilon_1\frac{E(s)}{E(0)} \leq \frac{2C}{t+1},\ t \geq 0. \tag{2.2}$$

By the definition of H_2, we have

$$\frac{t}{2}H_2'(\varepsilon_1\frac{E(t)}{E(0)})\varepsilon_1\frac{E(t)}{E(0)} \leq \int_{\frac{t}{2}}^t H_2'(\varepsilon_1\frac{E(s)}{E(0)})\varepsilon_1\frac{E(s)}{E(0)}ds.$$

Owing to (2.1), the above inequality becomes

$$H_2'(\varepsilon_1\frac{E(t)}{E(0)})\varepsilon_1\frac{E(t)}{E(0)} \leq \frac{2C}{t}. \tag{2.3}$$

If $t \geq 1$, then $t - \frac{1}{2} \geq \frac{t}{2}$. (2.3) becomes

$$H_2'(\varepsilon_1\frac{E(t)}{E(0)})\varepsilon_1\frac{E(t)}{E(0)} \leq \frac{2C}{t-\frac{1}{2}+\frac{1}{2}} \leq \frac{2C}{\frac{t}{2}+\frac{1}{2}} = \frac{4C}{t+1}. \tag{2.4}$$

If $0 < t < 1$,

$$H_2'(\varepsilon_1\frac{E(t)}{E(0)})\varepsilon_1\frac{E(t)}{E(0)} \leq C = \frac{C}{t+1}(t+1) \leq \frac{2C}{t+1}. \tag{2.5}$$

By (2.4) and (2.5), we get (2.2). Then, by the definition of $\widehat{H_2}(s)$,

$$\widehat{H_2}\left(\varepsilon_1\frac{E(t)}{E(0)}\right) \leq \frac{C}{t+1}$$

$$E(t) \leq C\widehat{H_2}^{-1}((t+1)^{-1})$$

Then, Theorem1.1 is proved.

References

[1] A. M. Al-Mahdi, M. M. Al-Gharabli, I. Kissami, A. Soufyane, and M. Zahri. Exponential and polynomial decay results for a swelling porous elastic system with a single nonlinear variable exponent damping: theory and numerics. *Z. Angew. Math. Phys.*, 74(2):Paper No. 72, 28, 2023.

[2] Fatiha Alabau-Boussouira and Kaïs Ammari. Sharp energy estimates for nonlinearly locally damped PDEs via observability for the associated undamped system. *J. Funct. Anal.*, 260(8):2424–2450, 2011.

[3] Tijani A. Apalara. General stability result of swelling porous elastic soils with a viscoelastic damping. *Z. Angew. Math. Phys.*, 71(6):Paper No. 200, 10, 2020.

[4] Tijani A. Apalara, Moruf O. Yusuf, and Babatunde A. Salami. On the control of viscoelastic damped swelling porous elastic soils with internal delay feedbacks. *J. Math. Anal. Appl.*, 504(2):Paper No. 125429, 12, 2021.

[5] Marcelo M. Cavalcanti, Flávio R. Dias Silva, and Valéria N. Domingos Cavalcanti. Uniform decay rates for the wave equation with nonlinear damping locally distributed in unbounded domains with finite measure. *SIAM J. Control Optim.*, 52(1):545–580, 2014.

[6] Marcelo M. Cavalcanti, Flávio R. Dias Silva, Valéria N. Domingos Cavalcanti, and André Vicente. Stability for the mixed problem involving the wave equation, with localized damping, in unbounded domains with finite measure. *SIAM J. Control Optim.*, 56(4):2802–2834, 2018.

[7] Marcelo Moreira Cavalcanti and Higidio Portillo Oquendo. Frictional versus viscoelastic damping in a semilinear wave equation. *SIAM J. Control Optim.*, 42(4):1310–1324, 2003.

[8] D. Ieşan. On the theory of mixtures of thermoelastic solids. *J. Thermal Stresses*, 14(4):389–408, 1991.

[9] Kun-Peng Jin, Jin Liang, and Ti-Jun Xiao. Uniform stability of semilinear wave equations with arbitrary local memory effects versus frictional dampings. *J. Differential Equations*, 266(11):7230–7263, 2019.

[10] Kun-Peng Jin, Jin Liang, and Ti-Jun Xiao. Asymptotic stability of energy for a weak viscoelastic plate equation with complementary frictional damping. *Appl. Math. Optim.*, 84(3):3025–3044, 2021.

[11] Muhammad I. Mustafa. On the control of dissipative viscoelastic Timoshenko beams. *Mediterr. J. Math.*, 18(2):Paper No. 49, 20, 2021.

[12] Muhammad I. Mustafa, Adel M. Al-Mahdi, and Mohammad M. Al-Gharabli. Theoretical decay results of a swelling soils system with frictional damping versus viscoelastic damping. *Mediterr. J. Math.*, 20(5):Paper No. 246, 21, 2023.

[13] Mitsuhiro Nakao. Decay of solutions of the wave equation with a local nonlinear dissipation. *Math. Ann.*, 305(3):403–417, 1996.

[14] R. Quintanilla. Exponential stability for one-dimensional problem of swelling porous elastic soils with fluid saturation. *J. Comput. Appl. Math.*, 145(2):525–533, 2002.

[15] A. J. A. Ramos, M. M. Freitas, D. S. Almeida, Jr., A. S. Noé, and M. J. Dos Santos. Stability results for elastic porous media swelling with nonlinear damping. *J. Math. Phys.*, 61(10):101505, 10, 2020.

[16] Jun-Min Wang and Bao-Zhu Guo. On the stability of swelling porous elastic soils with fluid saturation by one internal damping. *IMA J. Appl. Math.*, 71(4):565–582, 2006.

[17] Abderrahmane Youkana, Adel M. Al-Mahdi, and Salim A. Messaoudi. General energy decay result for a viscoelastic swelling porous-elastic system. *Z. Angew. Math. Phys.*, 73(3):Paper No. 88, 17, 2022.

[18] Enrike Zuazua. Exponential decay for the semilinear wave equation with locally distributed damping. *Comm. Partial Differential Equations*, 15(2):205–235, 1990.