UNDER PEER REVIEW

THE EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A CLASS OF QUASILINEAR ELLIPTIC EQUATIONS

Abstract: Let $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ is a bounded domain with a smooth C^2-boundary, $0 \in \partial\Omega$, and $\boldsymbol{n}$ denote the unit outward normal to $\partial\Omega$. We are concerned with the Neumann boundary problems:

$$\begin{cases} -div\left(|x|^{\alpha} \nabla u\right) + \lambda |x|^{\gamma} u = |x|^{\beta} u^{p(\alpha,\beta)-1}, & x \in \Omega \\ u(x) > 0, & x \in \Omega \\ \frac{\partial u}{\partial \boldsymbol{n}} = 0, & x \in \partial\Omega, \end{cases}$$

where $p(\alpha,\beta) = \frac{2(N+\beta)}{N-2+\alpha} > 2$, $\gamma > \alpha - 2$, $\alpha > 0$, $\beta < 0$. For certain region of the parameters α, β and γ, we establish the existence result of least energy solutions. Furthermore we obtain the multiplicity of solutions for a related linear perturbation problem.

Keywords: Caffarelli-Kohn-Nirenberg inequalities; Neumann Boundary; Least energy solutions.

1. **Introduction**

In this paper, we consider the following quasilinear elliptic problems with Neumann Boundary

$$\begin{cases} -div\left(|x|^{\alpha} |\nabla u|^{p-2} \nabla u\right) + \lambda |x|^{\gamma} u^{p-1} = |x|^{\beta} u^{p(\alpha,\beta)-1}, & x \in \Omega \\ u(x) > 0, & x \in \Omega \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \boldsymbol{n}} = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \in \mathbb{R}^N$ $(N \geqslant 3)$ is a bounded domain with C^2-boundary, $0 \in \partial\Omega$, $1 < p < N$ and $\alpha, \beta \in \mathbb{R}$ such that $p(\alpha,\beta) = \frac{p(N+\beta)}{N-p+\alpha} > p$. $\boldsymbol{n}$ denotes the unit outward normal to $\partial\Omega$. We recall the following well-known Caffarelli-Kohn-Nirenberg inequalities[3]

$$\left(\int_{R^N} |x|^{\beta} u^{p(\alpha,\beta)}\right)^{\frac{p}{p(\alpha,\beta)}} \leqslant S_{\alpha,\beta}^{-1} \int_{R^N} |x|^{\alpha} |\nabla u|^p, \qquad \forall u \in C_0^{\infty}\left(\mathbb{R}^N\right), \tag{1.2}$$

where

$$\alpha > -N + p,\ \frac{\alpha}{p} \geqslant \frac{\beta}{p(\alpha,\beta)},\ \beta \geqslant \alpha - p. \tag{1.3}$$

$p(\alpha,\beta)$ is called the critical Sobolev-Hardy exponent since when $\alpha = \beta = 0$ and $\alpha = 0,\ \beta = -p$, (1.2) are classical Sobolev and Hardy inequalities respectively. (1.2) has played an important role in many applications by virtue of the complete knowledge about the best constant $S_{\alpha,\beta}$ and the extremal functions.

Define the weighted Sobolev space $W^{1,p}_{\gamma,\alpha}(\Omega)$ by

$$W^{1,p}_{\gamma,\alpha}(\Omega) := \left\{ u \in L^p_{\gamma}(\Omega) : \int_{\Omega} |x|^{\alpha} |\nabla u|^p + \int_{\Omega} |x|^{\gamma} |u|^p < \infty \right\}$$

with the norm $\|u\|_{W^{1,p}_{\gamma,\alpha}(\Omega)} = \left(\int_{\Omega} |x|^{\alpha} |\nabla u|^p + \int_{\Omega} |x|^{\gamma} |u|^p\right)^{\frac{1}{p}}$, where $L^p_{\gamma}(\Omega)$ is the usual weighted $L^p(\Omega)$ space with the weight $|x|^{\gamma}$. From Lemma 2.1 and range of γ, we have $W^{1,p}_{\gamma,\alpha}(\Omega) \hookrightarrow$

$L^{p(\alpha,\beta)}(\Omega)$. But the embedding $W^{1,p}_{\gamma,\alpha}(\Omega) \hookrightarrow L^{p(\alpha,\beta)}(\Omega)$ is not compact. In addition, Consider the functional

$$J(u) = \frac{1}{p}\left(\int_\Omega |x|^\alpha |\nabla u|^p + |x|^\gamma |u|^p\right) - \frac{1}{p(\alpha,\beta)}\int_\Omega |x|^\beta u_+^{p(\alpha,\beta)}, u \in W^{1,p}_{\gamma,\alpha}(\Omega).$$

The functional $J(u)$ is well defined and is C^1-smooth in $W^{1,p}_{\gamma,\alpha}(\Omega)$, hence by the strong maximum principle (see Proposition 3.1 in [1]), its critical point is a weak solution to Problem (1.1). We will employ the minimax theory to find the nontrivial critical points of $J(u)$.

On the other hand let $\alpha = -pa$, $\beta = -bh$, then for $1 < p < N$, (1.1) is equivalent to

$$-div\left(|x|^{-pa}|\nabla u|^{p-2}\nabla u\right) = |x|^{-bh} u^{h-1} - \lambda |x|^\gamma u^{p-1}, u > 0, \tag{1.4}$$

where

$$-\infty < a < \frac{N-p}{p}, a - \frac{N-p}{p} < b < a+1, h = \frac{Np}{N-p(1+a+b)}.$$

which is related the classial Caffarelli-Kohn-Nirenberg inequalities:

$$\left\||x|^{-a}\nabla u\right\|_{L^p(\mathbb{R}^N)} \geqslant C_{a,b}\left\||x|^{-b} u\right\|_{L^h(\mathbb{R}^N)}, \qquad \forall u \in C_0^\infty(\mathbb{R}^N). \tag{1.5}$$

For inequality (1.5), according to the different range of parameters a and b, there is a quite extensive literature focusing on this type of problems. It can refer to the details from ([7], [10], [13]). Dolbeault et al. [5] proved an optimal result about the radial range of the extremal functions. Namely, $p \in (2, 2^*)$, assuming the integrability condition $\int_{R^N} |x|^{-bh} u^h dx < \infty$, the extremal functions of inequality (1.5) is radial either for

$$0 \leqslant a < \frac{N-2}{2} \ and \ a \leqslant b < a+1,$$

or for

$$a < 0 \ and \ b_{FS}(a) \leqslant b < a+1,$$

where

$$b_{FS}(a) := \frac{N\left(\frac{N-2}{2} - a\right)}{2\sqrt{\left(\frac{N-2}{2} - a\right)^2 + N - 1}} + \frac{2a + 2 - N}{2}.$$

The radial extremal functions of inequality (1.5) is

$$u_\varepsilon(x) = \frac{\varepsilon^{\frac{N-2+\alpha}{2(2-\alpha+\beta)}}}{\left(\varepsilon + |x|^{2-\alpha+\beta}\right)^{\frac{N-2+\alpha}{2-\alpha+\beta}}}. \tag{1.6}$$

In addition, we consider existence of the another quasilinear equation:

$$-div\left(|x|^\alpha |\nabla u|^{p-2}\nabla u\right) + |x|^\alpha u^{p-2} = a(x)|x|^\alpha |u|^{p-2} u + f(x), \tag{1.7}$$

where Ω is a bounded domain, $\alpha < 0$, $1 < p < N$. We suppose $a(x)$, $f(x)$, p, q meet the following conditions:

(i) $a(x) > 1$, $a(x) \to 1$ as $\text{dist}(x, \partial\Omega) \to 0$, $a(x) \in C(\Omega)$
(ii) $f(x) \geqslant 0$, $f(x) \not\equiv 0$, $f(x) \in W^{1,p}(\Omega)$
(iii) $p < q < p^* = \frac{Np}{N-p}$, $p < N$.

Theorem 1.1. *Let* $p = 2$, $\lambda > 0$, $\alpha - 1 < \gamma < \alpha$, $0 < \alpha < 2$, $\frac{\alpha-2}{2} < \beta < 0$, *and* (1.3) *hold. Then problem* (1.1) *has a least energy solution.*

Theorem 1.2. *If* $(i),(ii),(iii)$ *hold, there exists a constant* C*,* $\|f\|_{W^{1,p}_{\alpha}(\Omega)} \leqslant C$*, there exist at least two solutions to the problem* (1.7)*.*

Remark 1.3. In [1] where $p=2$, $0 \leqslant a < \frac{N-2}{2}$ and $a \leqslant b < a+1$, Bartsch et al. considered (1.1) in a cone or some other domains with Dirichlet or Neumann boundary conditions, and obtained some existence and nonexistence results of positive solutions. In [8], Bing-yu Kou popularised the above results to $1<p<N$. In this paper, we will consider the situation of $a<0$, namely $\alpha>0$. The range of radial solution $0<\alpha<2$, $\frac{\alpha-2}{2}<\beta<0$, equivalent to $-1<a<0$, $0<b<\frac{aN+N-2a^2-4a-2}{2(N-a-1)}<a+1$. The purpose is to change the range of parameters to prove that the solutions of the equation (1.1) still exists.

To prove our main results, we mainly apply the minimax argument in variational method. More precisely, we first construct a Palais-Smale sequence $\{u_i\}$ by using the Mountain Pass Lemma and then give a threshold value under which the Palais-Smale sequence is pre-compact. To verify that the energy corresponding to the Palais-Smale sequence $\{u_i\}$ is lower than the threshold, we use the function $u_\varepsilon(x)$ which can achieve $S_{\alpha,\beta}$ in $\mathbb{R}^N$ as a test function to estimate the energy $J(u_i)$.In the second section, we present the key lemmas and foundational estimates required for the proofs. By establishing and verifying a compactness condition, the proof of Theorem 1.1 is finalized in the third section.In addition, in the fourth section of this paper, we introduced another quasilinear equation. We mainly discussed existence of the equation solution under the condition of satisfying (i), (ii), (iii) in equation (1.7). For the method of combining equation (1.7) with perturbation, using the Ekeland variational principle and the Mountain pass theorem, first find a local minimum solution u_0 corresponding to the function near the zero. Then through the Mountain pass theorem, we can find a solution different from u_0.

2. **Preliminary results**

In this section, we will give some preliminary lemmas and some basic estimates which will be used to prove our main results. Define the weighted Sobolev space $W^{1,2}_{\gamma,\alpha}(\Omega)$ by

$$W^{1,2}_{\gamma,\alpha}(\Omega) := \left\{ u \in L^2_\gamma(\Omega) : \int_\Omega |x|^\alpha |\nabla u|^2 + \int_\Omega |x|^\gamma |u|^2 < \infty \right\}$$

provided with the norm

$$\|u\|_{W^{1,2}_{\gamma,\alpha}(\Omega)} = \left(\int_\Omega |x|^\alpha |\nabla u|^2 + \int_\Omega |x|^\gamma |u|^2 \right)^{\frac{1}{2}}.$$

Since $\partial\Omega$ is C^2-boundary, it is easy to know from Lemma 2.2 below that as $\alpha-2<\gamma\leqslant\alpha$, the embedding $W^{1,2}_{\gamma,\alpha}(\Omega) \hookrightarrow L^{p(\alpha,\beta)}_\beta(\Omega)$ is continuous. Hence (1.1) has a variational structure

$$J(u) = \frac{1}{2}\left(\int_\Omega |x|^\alpha |\nabla u|^2 + |x|^\gamma |u|^2 \right) - \frac{1}{p(\alpha,\beta)} \int_\Omega |x|^\beta u_+^{p(\alpha,\beta)}, u \in W^{1,2}_{\gamma,\alpha}(\Omega). \tag{2.1}$$

The following lemmas can be found in [1].

Lemma 2.1. *For* $1 \leq q < p(\alpha,\beta)$*, the imbedding* $W^{1,2}_{\gamma,\alpha}(\Omega) \hookrightarrow L^q_\beta(\Omega)$ *is compact.*

Lemma 2.2. *Let* $S_{\alpha,\beta}$ *be defined as in* (1.2)*. For all* $\delta>0$*, there exists a constant* $C(\delta)>0$ *depending on* δ *such that*

$$\left(\int_\Omega |x|^\beta u^{p(\alpha,\beta)} \right)^{\frac{2}{p(\alpha,\beta)}} \leq \left(2^{\frac{2+\beta-\alpha}{N+\beta}} S^{-1}_{\alpha,\beta} \right) \int_\Omega |x|^\alpha |\nabla u|^2 + C(\delta) \int_\Omega |x|^\alpha |u|^2 \tag{2.2}$$

for all $u \in W^{1,2}_{\gamma,\alpha}(\Omega)$*.*

Lemma 2.3. *Let* $(u_i) \subset W^{1,2}_{\gamma,\alpha}(\Omega)$ *satisfy* $J(u_i) \to c$ *and* $J'(u_i) \to 0$ *in* $\left(W^{1,2}_{\gamma,\alpha}(\Omega)\right)'$ *as* $i \to \infty$*. If*

$$c < \frac{\beta + 2 - \alpha}{4(N+\beta)} S_{\alpha,\beta}^{\frac{N+\beta}{\beta+2-\alpha}} \tag{2.3}$$

then (1.1) *has a solution* $u \in W^{1,2}_{\gamma,\alpha}(\Omega)$ *with* $u \neq 0$ *and* $J(u) \leq c$*.*

Suppose that $\pi_1, \cdots, \pi_{N-1}$ are the principal curvatures of $\partial\Omega$ at 0, so that the mean curvature is $\frac{1}{N-1}\sum_{k=1}^{N-1} \pi_k$.
Since $\partial\Omega$ is C^2-boundary at 0, the boundary near the origin can be represented by

$$x_N = h(x') = \frac{1}{2}\sum_{k=1}^{N-1} \pi_k x_k^2 + o\left(|x'|^2\right),$$

where $x'=(x_1, \cdots, x_{N-1}) \in D_\delta(0)$ for some $\delta > 0$, $D_\delta(0) = B_\delta(0) \cap \{x_N = 0\}$.
Set

$$K_1(\varepsilon) = \int_\Omega |x|^\alpha |\nabla u_\varepsilon|^2, K_2(\varepsilon) = \int_\Omega |x|^\beta u_\varepsilon^{p(\alpha,\beta)}, K_3(\varepsilon) = \int_\Omega |x|^\gamma u_\varepsilon^2,$$

where u_ε is given in (1.6).
Now let us estimate $K_1(\varepsilon)$, $K_2(\varepsilon)$, $K_3(\varepsilon)$ separately.

$$K_1(\varepsilon) = \int_{\mathbb{R}^N_+} |x|^\alpha |\nabla u_\varepsilon|^2 - \int_{(\mathbb{R}^N_+ \setminus \Omega)\setminus B(0,\delta)} |x|^\alpha |\nabla u_\varepsilon|^2 - \int_{B(0,\delta)} |x|^\alpha |\nabla u_\varepsilon|^2$$

and

$$\int_{\mathbb{R}^N_+} |x|^\alpha |\nabla u_\varepsilon|^2 = \frac{1}{2}(N-2+\alpha)^2 \int_{\mathbb{R}^N} \varepsilon^{\frac{N-2+\alpha}{2-\alpha+\beta}} \frac{|x|^{2+2\beta-\alpha}}{\left(\varepsilon + |x|^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}}} dx$$

$$= \frac{1}{2}(N-2+\alpha)^2 \int_{\mathbb{R}^N} \frac{|y|^{2+2\beta-\alpha}}{\left(\varepsilon + |y|^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}}} dy \tag{2.4}$$

where $\mathbb{R}^N_+ = \mathbb{R}^N \cap \{x_N > 0\}$, and

$$K_1 = (N-2+\alpha)^2 \int_{\mathbb{R}^N} \frac{|y|^{2+2\beta-\alpha}}{\left(\varepsilon + |y|^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}}} dy \tag{2.5}$$

is a constant independent of ε.

$$\int_{(\mathbb{R}^N_+ \setminus \Omega)\setminus B(0,\delta)} |x|^\alpha |\nabla u_\varepsilon|^2 dx = (N-2+\alpha)^2 \varepsilon^{\frac{N-2+\alpha}{2-\alpha+\beta}} \int_{(\mathbb{R}^N_+ \setminus \Omega)\setminus B(0,\delta)} \frac{|x|^{2+2\beta-\alpha}}{\left(\varepsilon + |x|^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}}} dx$$

$$= O\left(\varepsilon^{\frac{N-2+\alpha}{2-\alpha+\beta}}\right) \tag{2.6}$$

and

$$\int_{B(0,\delta)} |x|^\alpha |\nabla u_\varepsilon|^2 = \int_{D(0,\delta)} dx' \int_0^{g(x')} |x|^\alpha |\nabla u_\varepsilon|^2 dx_N + \int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} |x|^\alpha |\nabla u_\varepsilon|^2 dx_N.$$

Observing that

$$A_\varepsilon = \int_{D(0,\delta)} dx' \int_0^{g(x')} |x|^\alpha |\nabla u_\varepsilon|^2 \, dx_N$$
$$= (N-2+\alpha)^2 \int_{D(0,\delta)} dy' \int_0^{\varepsilon^{\frac{1}{2-\alpha+\beta}} g(y')} \frac{|y|^{2+2\beta-\alpha}}{\left(1+|y|^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}}} dy_N$$
$$= (N-2+\alpha)^2 \varepsilon^{\frac{1}{2-\alpha+\beta}} \int_{D(0,\delta)} \frac{|y|^{2+2\beta-\alpha} g(y')}{\left(1+|y|^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}}} dy', \tag{2.7}$$

and

$$\int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} |x|^\alpha |\nabla u_\varepsilon|^2 \, dx_N$$
$$= (N-2+\alpha)^2 \varepsilon^{\frac{N-2+\alpha}{2-\alpha+\beta}} \int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} \frac{|x|^{2+2\beta-\alpha}}{\left(\varepsilon+|x|^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}}} dx_N$$
$$= (N-2+\alpha)^2 \varepsilon^{\frac{N-2+\alpha}{2-\alpha+\beta}} \int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} \frac{\left(|x'|^2+x_N^2\right)^{\frac{2+\beta-\alpha}{2}} \left(|x'|^2+x_N^2\right)^{\frac{\beta}{2}}}{\left(\varepsilon+\left(|x'|^2+x_N^2\right)^{\frac{2+\beta-\alpha}{2}}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}}} dx_N$$
$$\leqslant (N-2+\alpha)^2 \varepsilon^{\frac{N-2+\alpha}{2-\alpha+\beta}} \int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} \frac{\left(|x'|^2+x_N^2\right)^{\frac{\beta}{2}}}{\left(\varepsilon+\left(|x'|^2+x_N^2\right)^{\frac{2+\beta-\alpha}{2}}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}-1}} dx_N$$
$$\leqslant (N-2+\alpha)^2 \varepsilon^{\frac{N-2+\alpha}{2-\alpha+\beta}} \int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} \frac{|x'|^\beta}{\left(\varepsilon+|x'|^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}-1}} dx_N$$
$$= (N-2+\alpha)^2 \varepsilon^{\frac{N-2+\alpha}{2-\alpha+\beta}} \int_{D(0,\delta)} \frac{|x'|^\beta |h(x')-g(x')|}{\left(\varepsilon+|x'|^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}-1}} dx'.$$

Since $h(x') = g(x') + o\left(|x'|^2\right)$, it follows that $\forall\varepsilon > 0$, $\exists C(\sigma) > 0$ such that $h(x') - g(x') \leqslant \sigma |x'|^2 + C(\sigma) |x'|^{\frac{5}{2}}$ for $x' \in D(0,\delta)$.
Therefore

$$\int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} |x|^\alpha |\nabla u_\varepsilon|^2 \, dx_N$$
$$\leqslant (N-2+\alpha)^2 \varepsilon^{\frac{N-2+\alpha}{2-\alpha+\beta}} \int_{D(0,\delta)} \frac{|x'|^\beta \left(\sigma |x'|^2 + C(\sigma) |x'|^{\frac{5}{2}}\right)}{\left(\varepsilon+|x'|^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}-1}} dx'$$

$$= (N-2+\alpha)^2 \varepsilon^{\frac{1}{2-\alpha+\beta}} \int_{D(0,\delta)} \frac{|y'|^{\beta+2}\left(\sigma + C(\sigma)\varepsilon^{\frac{1}{2(2-\alpha+\beta)}} |y'|^{\frac{1}{2}}\right)}{\left(1+|y'|^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}-1}} dy'$$

$$\leqslant C\varepsilon^{\frac{1}{2-\alpha+\beta}}\left(\sigma + C(\sigma)\varepsilon^{\frac{1}{2(2-\alpha+\beta)}}\right),$$

which implies

$$\int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} |x|^\alpha |\nabla u_\varepsilon|^2 \, dx_N = o\left(\varepsilon^{\frac{1}{2-\alpha+\beta}}\right) \qquad as \qquad \varepsilon \to 0. \tag{2.8}$$

Thus from $(2.5)-(2.8)$, we obtain

$$K_1(\varepsilon) = \frac{1}{2}K_1 - A_\varepsilon + o\left(\varepsilon^{\frac{1}{2-\alpha+\beta}}\right). \tag{2.9}$$

On the other hand, similary to esumate of $K_1(\varepsilon)$ we have

$$K_2(\varepsilon) = \int_{\mathbb{R}^N_+} |x|^\beta {u_\varepsilon}^{p(\alpha,\beta)} - \int_{\left(\mathbb{R}^N_+\setminus\Omega\right)\setminus B(0,\delta)} |x|^\beta {u_\varepsilon}^{p(\alpha,\beta)}$$

$$-\int_{D(0,\delta)} dx' \int_0^{g(x')} |x|^\beta {u_\varepsilon}^{p(\alpha,\beta)} dx_n + \int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} |x|^\beta {u_\varepsilon}^{p(\alpha,\beta)} dx_n.$$

With similar calculations we obtain

$$\int_{\mathbb{R}^N_+} |x|^\beta {u_\varepsilon}^{p(\alpha,\beta)} = \frac{1}{2}\int_{\mathbb{R}^N} \frac{|y|^\beta}{\left(1+|y|^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}}} dy \tag{2.10}$$

is a constant independent of ε, hence we remember $\int_{\mathbb{R}^N} \frac{|y|^\beta}{\left(1+|y|^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}}} dy$ as K_2,

and

$$\int_{\left(\mathbb{R}^N_+\setminus\Omega\right)\setminus B(0,\delta)} |x|^\beta {u_\varepsilon}^{p(\alpha,\beta)} = O\left(\varepsilon^{\frac{N+\beta}{2-\alpha+\beta}}\right). \tag{2.11}$$

Similar that

$$B_\varepsilon = \int_{D(0,\delta)} dx' \int_0^{g(x')} |x|^\beta u_\varepsilon^{p(\alpha,\beta)} dx_N = \varepsilon^{\frac{1}{2-\alpha+\beta}} \int_{D(0,\delta)} \frac{|y'|^\beta g(y')}{\left(1+|y'|^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}}} dy', \tag{2.12}$$

and

$$\int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} |x|^\beta {u_\varepsilon}^{p(\alpha,\beta)} dx_N$$

$$= \varepsilon^{\frac{N+\beta}{2-\alpha+\beta}} \int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} \frac{\left(x_N^2 + |x'|^2\right)^{\frac{\beta}{2}}}{\left(\varepsilon + \left(x_N^2+|x'|^2\right)^{\frac{2-\alpha+\beta}{2}}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}}} dx_N$$

$$\leqslant \varepsilon^{\frac{N+\beta}{2-\alpha+\beta}} \int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} \frac{|x'|^\beta}{\left(\varepsilon + |x'|^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}}} dx_N$$

$$\leqslant \varepsilon^{\frac{N+\beta}{2-\alpha+\beta}} \int_{D(0,\delta)} \frac{|x'|^{\beta} |h(x') - g(x')|}{\left(\varepsilon + |x'|^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}}} dx'$$

$$= \varepsilon^{\frac{1}{2-\alpha+\beta}} \int_{D(0,\delta)} \frac{|y'|^{\beta+2} \left(\sigma + C(\sigma) |y'|^{\frac{1}{2}} \varepsilon^{\frac{1}{2(2-\alpha+\beta)}}\right)}{\left(\varepsilon + |x'|^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}}} dy'$$

$$\leqslant C\varepsilon^{\frac{1}{2-\alpha+\beta}} \left(\sigma + C(\sigma) \varepsilon^{\frac{1}{2(2-\alpha+\beta)}}\right).$$

Hence,

$$\int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} |x|^{\beta} u_{\varepsilon}{}^{p(\alpha,\beta)} dx_N = o\left(\varepsilon^{\frac{1}{2-\alpha+\beta}}\right). \tag{2.13}$$

Thus from $(2.10) - (2.13)$, we obtain

$$K_2(\varepsilon) = \frac{1}{2} K_2 - B_{\varepsilon} + o\left(\varepsilon^{\frac{1}{2-\alpha+\beta}}\right). \tag{2.14}$$

From the range of γ, it is easy to get

$$K_3(\varepsilon) = \int_{\Omega} |x|^{\gamma} u_{\varepsilon}^2 = O\left(\varepsilon^{\frac{2-\alpha+\gamma}{2-\alpha+\beta}}\right). \tag{2.15}$$

Moreover, from Caffarelli-Kohn-Nirenberg inequalities K_1, K_2 satisfy

$$\frac{K_1}{K_2^{\frac{N-2+\alpha}{N+\beta}}} = S_{\alpha,\beta}.$$

3. The existence of least energy solutions

Set

$$c^* = \inf_{u} \sup_{t>0} J(tu).$$

where the infimum is taken over all u $\in W^{1,2}_{\gamma,\alpha}(\Omega)$ with u $\geqslant 0$, u $\neq 0$. It is easy to check that $c^* \geqslant c$, where c is the mountain-pass level defined as

$$c = \inf_{\psi \in \Psi} \sup_{t \in (0,1)} J(\psi(t)),$$

with $\Psi = \left\{\psi \in C\left([0,1], W^{1,2}_{\gamma,\alpha}(\Omega)\right) : \psi(0) = 0, \psi(1) \neq 0, J(\psi(1)) \leqslant 0\right\}$. Now we check that

$$c^* < \frac{\beta+2-\alpha}{4(N+\beta)} S_{\alpha,\beta}^{\frac{N+\beta}{\beta+2-\alpha}}.$$

Consider

$$u_{\varepsilon}(x) = \frac{\varepsilon^{\frac{N-2+\alpha}{2(2-\alpha+\beta)}}}{\left(\varepsilon + |x|^{2-\alpha+\beta}\right)^{\frac{N-2+\alpha}{2-\alpha+\beta}}}.$$

In the follow we will proof that for ε small enough there holds

$$\sup_{t>0} J(tu) < \frac{\beta+2-\alpha}{4(N+\beta)} S_{\alpha,\beta}^{\frac{N+\beta}{\beta+2-\alpha}}. \tag{3.1}$$

As a consequence, $\sup_{t>0} J(tu_\varepsilon)$ will be attained by a bounded t_ε for ε small enough. Thus,

$$\sup_{t>0} J(tu_\varepsilon) \leqslant \sup_{t>0}\left[\frac{1}{2}K_1(\varepsilon)t^2 - \frac{K_2(\varepsilon)}{p(\alpha,\beta)}t^{p(\alpha,\beta)}\right] + o\left(\varepsilon^{\frac{1}{2-\alpha+\beta}}\right),$$

where $t = \left(\frac{K_1(\varepsilon)}{K_2(\varepsilon)}\right)^{\frac{N-2+\alpha}{2(\beta+2-\alpha)}}$, we obtain

$$\sup_{t>0}\left[\frac{1}{2}K_1(\varepsilon)t^2 - \frac{K_2(\varepsilon)}{p(\alpha,\beta)}t^{p(\alpha,\beta)}\right] = \frac{\beta+2-\alpha}{2(N+\beta)}\left(\frac{K_1(\varepsilon)}{K_2(\varepsilon)^{\frac{N-2+\alpha}{N+\beta}}}\right)^{\frac{N+\beta}{\beta+2-\alpha}}.$$

Hence

$$\sup_{t>0} J(tu_\varepsilon) \leqslant \frac{\beta+2-\alpha}{2(N+\beta)}\left(\frac{K_1(\varepsilon)}{K_2(\varepsilon)^{\frac{N-2+\alpha}{N+\beta}}}\right)^{\frac{N+\beta}{\beta+2-\alpha}},$$

so it suffices to prove

$$\frac{\beta+2-\alpha}{2(N+\beta)}\left(\frac{K_1(\varepsilon)}{K_2(\varepsilon)^{\frac{N-2+\alpha}{N+\beta}}}\right)^{\frac{N+\beta}{\beta+2-\alpha}} < \frac{\beta+2-\alpha}{4(N+\beta)}S_{\alpha,\beta}^{\frac{N+\beta}{\beta+2-\alpha}}.$$

Through the equivalent transformation, which is equivalent to

$$K_1(\varepsilon)\left(\frac{K_2}{2}\right)^{\frac{N-2+\alpha}{N+\beta}} < \frac{K_1}{2}\left(K_2(\varepsilon)^{\frac{N-2+\alpha}{N+\beta}}\right), \tag{3.2}$$

and from (2.9) and (2.14),

$$\begin{aligned}
&\left(\frac{1}{2}K_1 - A_\varepsilon\right)\left(\frac{K_2}{2}\right)^{\frac{N-2+\alpha}{N+\beta}} \\
&< \frac{K_1}{2}\left(\frac{1}{2}K_2 - B_\varepsilon + o\left(\varepsilon^{\frac{1}{2-\alpha+\beta}}\right)\right)^{\frac{N-2+\alpha}{N+\beta}} + o\left(\varepsilon^{\frac{1}{2-\alpha+\beta}}\right) \\
&= \frac{K_1}{2}\left(\left(\frac{1}{2}K_2\right)^{\frac{N-2+\alpha}{N+\beta}} - \frac{N-2+\alpha}{N+\beta}\left(\frac{1}{2}K_2\right)^{\frac{\alpha-2-\beta}{N+\beta}}B_\varepsilon\right) + o\left(\varepsilon^{\frac{1}{2-\alpha+\beta}}\right),
\end{aligned}$$

so (3.2) is equivalent to

$$\frac{A_\varepsilon}{B_\varepsilon} > \frac{N-2+\alpha}{N+\beta}\frac{K_1}{K_2} + o(1).$$

From (2.7) and (2.12), and through polar coordinates we have

$$\lim_{\varepsilon\to 0}\frac{A_\varepsilon}{B_\varepsilon} = (N-2+\alpha)^2\frac{\int_0^\infty \frac{r^{N+2+2\beta-\alpha}}{\left(1+r^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}}}dr}{\int_0^\infty \frac{r^{N+\beta}}{\left(1+r^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}}}dr}, \tag{3.3}$$

and

$$\frac{N-2+\alpha}{N+\beta}\frac{K_1}{K_2} = \frac{(N-2+\alpha)^3}{(N+\beta)}\frac{\int_0^\infty \frac{r^{N+1+2\beta-\alpha}}{\left(1+r^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}}}dr}{\int_0^\infty \frac{r^{N-1+\beta}}{\left(1+r^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}}}dr}. \tag{3.4}$$

Integrating by parts, we obtain for $2 \leqslant k < 2N+\alpha+\beta-1$

$$\begin{aligned}&\int_0^\infty \frac{r^{k-2}}{\left(1+r^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}-1}}dr\\&=\frac{2N+\alpha+\beta-2}{k-1}\int_0^\infty \frac{r^{k-\alpha+\beta}}{\left(1+r^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}}}dr\\&=\frac{2N+\alpha+\beta-2}{k-1}\left(\int_0^\infty \frac{r^{k-2}}{\left(1+r^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}-1}}dr-\int_0^\infty \frac{r^{k-2}}{\left(1+r^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}}}dr\right).\end{aligned}$$

Hence,

$$\int_0^\infty \frac{r^{k-2}}{\left(1+r^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}}}dr=\frac{2N+\alpha+\beta-1-k}{k-1}\int_0^\infty \frac{r^{k-\alpha+\beta}}{\left(1+r^{2-\alpha+\beta}\right)^{\frac{2(N+\beta)}{2-\alpha+\beta}}}dr.$$

Now choosing $k=N+2+\beta$, one gets

$$\lim_{\varepsilon\to 0}\frac{A_\varepsilon}{B_\varepsilon}=\frac{(N-2+\alpha)^2\,(N+\beta+1)}{N-3+\alpha}.$$

On the other hand, choosing $k=N+1+\beta$, one gets

$$\frac{N-2+\alpha}{N+\beta}\frac{K_1}{K_2}=(N-2+\alpha)^2\,.$$

Since $\beta>\alpha-2$, thus (3.1) has been proved to be true. From Lemma 2.3, Theorem 1.1 is completed.

4. **The existence of the solution**

In this section, we consider the following problem

$$-div\left(|x|^\alpha |\nabla u|^{p-2}\nabla u\right)+|x|^\alpha u^{p-2}=a(x)\,|x|^\alpha |u|^{p-2}u+f(x)\,,$$

where Ω is a bounded domain, $\alpha<0$, $1<p<N$.

Define the weighted Sobolev space $W_\alpha^{1,p}(\Omega)$ by

$$W_\alpha^{1,p}(\Omega)=\left\{u\in L_\alpha^p(\Omega):\int_\Omega |x|^\alpha |\nabla u|^p+\int_\Omega |x|^\alpha |u|^p<\infty\right\}$$

provided with the norm $\|u\|_{W_\alpha^{1,p}(\Omega)}=\left(\int_\Omega |x|^\alpha |\nabla u|^p+\int_\Omega |x|^\alpha |u|^p\right)^{\frac{1}{p}}$.

We say that $u(x)$ is the weak solution of the problem (4.1), which means that for all $\varphi\in C_0^\infty(\Omega)$ such that

$$\int_\Omega |x|^\alpha |\nabla u|^{p-2}\nabla u\nabla\varphi dx+\int_\Omega |x|^\alpha |u|^{p-2}u\varphi dx=\int_\Omega a(x)\,|x|^\alpha |u|^{q-2}u\varphi dx+\int_\Omega f(x)\,\varphi dx.$$

Define functional

$$I_f(u)=\frac{1}{p}\left(\int_\Omega |x|^\alpha |\nabla u|^p+|x|^\alpha |u|^p\,dx\right)-\frac{1}{q}\int_\Omega a(x)\,|x|^\alpha |u|^q\,dx-\int_\Omega f(x)\,udx. \tag{4.1}$$

It is easy to verify that the nontrivial critical points of the functional (4.1) correspond to the solutions of (1.7). And define

$$I(u)=\frac{1}{p}\left(\int_\Omega |x|^\alpha |\nabla u|^p+|x|^\alpha |u|^p\,dx\right)-\frac{1}{q}\int_\Omega a(x)\,|x|^\alpha |u|^q\,dx, \tag{4.2}$$

$$I^{\infty}(u)=\frac{1}{p}\left(\int_{\Omega}|x|^{\alpha}|\nabla u|^{p}+|x|^{\alpha}|u|^{p}\,dx\right)-\frac{1}{q}\int_{\Omega}|x|^{\alpha}|u|^{q}\,dx, \tag{4.3}$$

$$I^{\infty}=inf\left\{I^{\infty}(u)\mid u\in W_{\alpha}^{1,p}(\Omega),u\geqslant 0,u\neq 0,F(u)=1\right\},$$

where $F(u)$ which is defined by

$$F(u)=\begin{cases}0, & u\equiv 0\\ \frac{\int_{\Omega}|x|^{\alpha}|\nabla u|^{p}+|x|^{\alpha}|u|^{p}dx}{\int_{\Omega}|x|^{\alpha}|u|^{q}dx}, & u\neq 0\end{cases}.$$

Set $\bar{u}$ is the extremal function of (4.4), and $\bar{u}$ is satisfied

$$\bar{u}\in W_{\alpha}^{1,p}(\Omega),\ \bar{u}\geqslant 0,\ \bar{u}\neq 0,\ I^{\infty}=\sup_{t>0}I^{\infty}(t\bar{u}). \tag{4.4}$$

Lemma 4.1. *Let (u_m) be a sequence in $W_{\alpha}^{1,p}(\Omega)$ satisfying $u_m\rightharpoonup u_0$, the following conclusions will be established:*

$$\lim_{m\to\infty}\int_{\Omega}|x|^{\alpha}|\nabla u_m|^{p}\,dx-\int_{\Omega}|x|^{\alpha}|\nabla u_0|^{p}\,dx=\lim_{m\to\infty}\int_{\Omega}|x|^{\alpha}|\nabla(u_m-u_0)|^{p}\,dx,$$

$$\lim_{m\to\infty}\int_{\Omega}|x|^{\alpha}|u_m|^{p}\,dx-\int_{\Omega}|x|^{\alpha}|u_0|^{p}\,dx=\lim_{m\to\infty}\int_{\Omega}|x|^{\alpha}|u_m-u_0|^{p}\,dx.$$

Lemma 4.2. *For $(u_m)\subset W_{\alpha}^{1,p}(\Omega)$, $u_m\rightharpoonup u_0$, and $I_f(u)$ satisfies the (PS) condition, then u_0 is a critical value of $I_f(u)$.*

Lemma 4.3. *Let (u_m) be a $(PS)_c$ sequence in $W_{\alpha}^{1,p}(\Omega)$, $u_0\in W_{\alpha}^{1,p}(\Omega)$, and $u_m\rightharpoonup u_0$, then either $u_m\to u_0$, $I_f(u_0)=c$, or $c\geqslant I_f(u_0)+I^{\infty}$.*

Proof. Set $B_R=\left\{u\in W_{\alpha}^{1,p}(\Omega),\|u\|<R\right\}$,

$$I_0=I_0(R)=\inf_{u\in B_R}I_f(u). \tag{4.5}$$

Since (u_m) be a $(PS)_c$ sequence in $W_{\alpha}^{1,p}(\Omega)$, and $u_m\rightharpoonup u_0$, let $v_m=u_m-u_0$, we have $v_m\rightharpoonup 0$. From lemma 4.1 we can obtain

$$\begin{aligned}I(v_m)&=I(u_m-u_0)\\&=\frac{1}{p}\int_{\Omega}|x|^{\alpha}|\nabla(u_m-u_0)|^{p}+|x|^{\alpha}|u_m-u_0|^{p}\,dx\\&-\frac{1}{q}\int_{\Omega}a(x)|x|^{\alpha}|u_m-u_0|^{q}\,dx,\end{aligned}$$

and

$$\left\langle I'(u_m),u_m\right\rangle=\int_{\Omega}|x|^{\alpha}|\nabla u_m|^{p}+|x|^{\alpha}|u_m|^{p}\,dx-\int_{\Omega}a(x)|x|^{\alpha}|u_m|^{q}\,dx,$$

$$I_f(u_0)+I(v_m)=c+o(1), \tag{4.6}$$

$$\left\langle I'(u_m),u_m\right\rangle=\left\langle I'(v_m),v_m\right\rangle+o(1). \tag{4.7}$$

If $v_m\to 0$, then $u_m\to u_0$, and $I_f(u_0)=\lim_{m\to\infty}I_f(u_m)=c$. Now we assume that $\|v_m\|\to\eta>0$ as $m\to\infty$, then

$$\int_{\Omega}(a(x)-1)|x|^{\alpha}|v_m|^{p}\,dx$$

$$= \int_{\overline{B_R(0)}} (a(x)-1)|x|^{\alpha}|v_m|^p\,dx + \int_{\Omega\backslash\overline{B_R(0)}} (a(x)-1)|x|^{\alpha}|v_m|^p\,dx$$
$$\leqslant 2\,\underset{\overline{B_R(0)}}{sup}\,a(x)\int_{\overline{B_R(0)}} |x|^{\alpha}|v_m|^p\,dx + \underset{\Omega\backslash\overline{B_R(0)}}{sup}(a(x)-1)\int_{\Omega}|x|^{\alpha}|v_m|^p\,dx.$$

Through a simple calculation, we can get

$$\int_{\Omega}|x|^{\alpha}|\nabla v_m|^p\,dx = \sigma_m^{n-p}\int_{\Omega}|\sigma_m x|^{\alpha}|\nabla\xi_m|^p\,dx, \tag{4.8}$$

and

$$\int_{\Omega}|x|^{\alpha}|v_m|^p\,dx = \sigma_m^{n}\int_{\Omega}|\sigma_m x|^{\alpha}|\xi_m|^p\,dx. \tag{4.9}$$

From (4.7), (4.8), and(4.9), we can obtain

$$\sigma_m^{n-p}\int_{\Omega}|x|^{\alpha}|\nabla\xi_m|^p\,dx+\sigma_m^{n}\int_{\Omega}|x|^{\alpha}|\xi_m|^p\,dx-\sigma_m^{n}\int_{\Omega}|x|^{\alpha}|\xi_m|^q\,dx-\int_{\Omega}(a(x)-1)|x|^{\alpha}|v_m|^q\,dx = o(1).$$

Then let $\varepsilon_m = \sigma_m^{-n}\left[\int_{\Omega}(a(x)-1)|x|^{\alpha}|v_m|^q\,dx + o(1)\right]$, we have

$$\int_{\Omega}|x|^{\alpha}\left(|\nabla\xi_m|^p+|\xi_m|^p-|\xi_m|^q\right)dx = \sigma_m^{-n}\left(\sigma_m^p-1\right)\int_{\Omega}|x|^{\alpha}|\nabla\xi_m|^p\,dx+\varepsilon_m. \tag{4.10}$$

We can take σ_m to make $\xi_m \in \left\{u\in W_{\alpha}^{1,p}(\Omega)\,|F(u)=1\right\}$. From (4.10), we can obtain $\sigma_m\to 1$ as $m\to\infty$. Then calculate $I(v_m)$, we have the following estimates,

$$\begin{aligned}
I(v_m) &= \frac{1}{p}\int_{\Omega}|x|^{\alpha}|\nabla v_m|^p+|x|^{\alpha}|v_m|^p\,dx-\frac{1}{q}\int_{\Omega}a(x)|x|^{\alpha}|v_m|^q\,dx\\
&= \frac{1}{p}\sigma_m^{n-p}\int_{\Omega}|\sigma_m x|^{\alpha}|\nabla\xi_m|^p\,dx+\sigma_m^{n}\int_{\Omega}|\sigma_m x|^{\alpha}|\xi_m|^p\,dx-\frac{1}{q}\int_{\Omega}a(x)|x|^{\alpha}|v_m|^q\,dx\\
&= \sigma_m^n\left(\frac{1}{p}\int_{\Omega}|\sigma_m x|^{\alpha}|\nabla\xi_m|^p+|\sigma_m x|^{\alpha}|\xi_m|^p\,dx-\frac{1}{q}\int_{\Omega}a(x)|\sigma_m x|^{\alpha}|\xi_m|^q\,dx\right)\\
&+\frac{1}{p}\left(1-\sigma_m^p\right)\int_{\Omega}|x|^{\alpha}|\nabla v_m|^p\,dx\\
&= \sigma_m^n I^{\infty}(\xi_m)+\frac{1}{p}\sigma_m^p\left(\sigma_m^{-p}-1\right)\int_{\Omega}|x|^{\alpha}|\nabla v_m|^p\,dx+o(1)\\
&\geqslant \sigma_m^n I^{\infty}+\frac{1}{p}\sigma_m^p\left(\sigma_m^{-p}-1\right)\int_{\Omega}|x|^{\alpha}|\nabla v_m|^p\,dx+o(1).
\end{aligned}$$

Hence $\sigma_m\to 1$, then we have $I(v_m)\geqslant I^{\infty}+o(1)$. From (4.6), it suffices to prove $c\geqslant I_f(u_0)+I^{\infty}$. □

Lemma 4.4. *Let $g\in\left(W_{\alpha}^{1,p}(\Omega)\right)^*$ be nonnegative satisfying $\|g\|_{\left(W_{\alpha}^{1,p}(\Omega)\right)^*}\leqslant C$. Then for $\varepsilon>0$, there is u_0 in the $W_{\alpha}^{1,p}(\Omega)$ satisfying $I_0(\varepsilon)=I_f(u_0)$, and $I_f'(u_0)=0$.*

Proof. For all $u>0$ in $W_{\alpha}^{1,p}(\Omega)$, we have

$$I_f(tu)=\frac{1}{p}t^p\|u\|^p-\frac{1}{q}t^q\int_{\Omega}a(x)|x|^{\alpha}|u|^q\,dx-t\int_{\Omega}g(x)u\,dx,$$

and

$$\frac{dI_f(tu)}{dt}=t^{p-1}\|u\|^p-t^{q-1}\int_{\Omega}a(x)|x|^{\alpha}|u|^q\,dx-\int_{\Omega}g(x)u\,dx.$$

So there exists a constant t_1, it follows that $t < t_1$ such that $\frac{dI_f(tu)}{dt} < 0$. And because of $I_f(0) = 0$, it exists $R > 0$, we have $I_0(R) < 0$. Consider $\|g(x)\|_{\left(W_\alpha^{1,p}(\Omega)\right)^*} \leqslant C$, we will get that for C small enough there exists another t such that $I_f(tu)$ is a monotonically increasing function. Then for $\forall$ $u \in W_\alpha^{1,p}(\Omega)$, $I_f(tu)$ has a local minial value. Through the Ekeland variational principle, we can get a $(PS)_c$ sequence $\{u_m\}$ of $I_f(u)$ satisfy the following conditions:

$$I_f(u_m) = I_0(R) + o(1), I_f'(u_m) \to 0\,(m \to \infty)$$

Because of $I_0(R) < 0$, there exists $u_0 \in W_\alpha^{1,p}(\Omega)$ such that $u_m \rightharpoonup u_0$. Thus we have $I_0(R) = I_0(u_0)$, $I_f'(u_0) = 0$. □

Set $\overline{u} \in W_\alpha^{1,p}(\Omega)$ is the function of (4.5). We have $I(t\overline{u}) \to -\infty\,(t \to \infty)$. Then we set the following definition:

$$\Gamma = \left\{\gamma \in C\left([0,1], W_\alpha^{1,p}(\Omega)\right) : \gamma(0) = 0, \gamma(1) = \overline{t}\overline{u}\right\}.$$

$$C_f = \underset{\gamma \in \Gamma}{inf} \underset{u \in \gamma}{sup} I_f(u)$$

Lemma 4.5. *For constant $C > 0$, for $R_0 > 0$ small enough, if $\|f\|_{\left(W_\alpha^{1,p}(\Omega)\right)^*} \leqslant C$, there holds $C_f < I_0 + I^\infty$, where $I^\infty = I(\overline{u})$, $I_0 = \underset{u \in B_{R_0}}{inf} I_f(u)$.*

Proof. Consider $a(x) > 1$, $a(x) \to 1$ as $|x| \to \partial\Omega$, for $\forall$ $t > 0$, $I(t\overline{u}) < I^\infty(t\overline{u})$, there exists $t_0 \in (0,t)$ to satisfy the following formulas:

$$\underset{t \geqslant 0}{sup} I(t\overline{u}) = I(t_0\overline{u}) < I^\infty(t_0\overline{u}) \leqslant \underset{t \geqslant 0}{sup} I^\infty(t\overline{u}) = I^\infty.$$

Thus we can find $\varepsilon_0 > 0$ to make $\underset{t \geqslant 0}{sup} I(t\overline{u}) < I^\infty - \varepsilon_0$. On the other hand, for the above ε_0, $\|f\|_{\left(W_\alpha^{1,p}(\Omega)\right)^*} \leqslant C_1$, $\exists$ $C_1 > 0$, we can obtain:

$$\left|\underset{u \in B_{R_0}}{inf} I_f(u)\right| = |I_0(R)| \leqslant \frac{\varepsilon_0}{2}.$$

And for $\forall$ $u \in \left\{\gamma_0 = t\overline{t}\overline{u} \middle| 0 \leqslant t \leqslant 1\right\}$, we have

$$|I_f(u) - I(u)| = \left|\int_\Omega fudx\right| \leqslant t\overline{t}\left|\int_\Omega f\overline{u}dx\right| \leqslant \overline{t}\,\|\overline{u}\|\,\|f\|_{\left(W_\alpha^{1,p}(\Omega)\right)^*}.$$

Since $\exists$ $C > 0$, $\|f\|_{\left(W_\alpha^{1,p}(\Omega)\right)^*} < C$, thus it suffices to prove $|I_f(u) - I(u)| < \frac{\varepsilon_0}{2}$. In summary we can get

$$C_f \leqslant \underset{u \in \gamma_0}{sup} I_f(u) \leqslant \underset{u \in \gamma_0}{sup} I(u) + \frac{\varepsilon_0}{2} = \underset{t \geqslant 0}{sup} I(t\overline{u}) + \frac{\varepsilon_0}{2} < I^\infty - \frac{\varepsilon_0}{2} < I^\infty + I_0.$$

□

Now we are ready to give the proof of Theorem 4.2.

Proof of Theorem 4.2. First from Lemma 4.4, it is easy to get $u_0 \in W_\alpha^{1,p}(\Omega)$, u_0 is the solution of the problem (1.7). In addition $\exists$ $C > 0$, $R > 0$, $\|f\|_{\left(W_\alpha^{1,p}(\Omega)\right)^*} \leqslant C$, we have $I_f(R) > 0$.Thus the conditions of Mountain pass theorem are established, we can get that $\{u_m\}$ is the $(PS)_{C_f}$ sequence of I_f, where $\{u_m\} \subset W_\alpha^{1,p}(\Omega)$. And it is easy to proof that $\{u_m\}$ is bounded in $W_\alpha^{1,p}(\Omega)$, hence there exists a subsequence $\left\{u_{m_j}\right\}$ such that $u_{m_j} \rightharpoonup u_1$, where $u_1 \in W_\alpha^{1,p}(\Omega)$. Now from Lemma 4.2,

$$I_f'(u_1) = 0,$$

so that u_1 is the weak solution of (4.1). Then we will proof $I_f'(u_1) = 0$. From the Lemma 4.3, either we have $u_{m_j} \to u_1$, or $I_f(u_1) = \lim_{j\to\infty} I_f\left(u_{m_j}\right) = C_f \geqslant I^\infty + I_0$. And from Lemma 4.5, we have $C_f < I^\infty + I_0$, hence $u_{m_j} \to u_1$. Now we can obtain

$$I_f(u_1) = \lim_{j\to\infty} I_f\left(u_{m_j}\right) = C_f > 0 > I_f(u_0).$$

This shows that u_0 and u_1 are different solutions of (1.7).

□

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