SPECTRAL INTEGRAL VARIATION OF GRAPH THEORY

ABSTRACT

The topic of spectral integral variation in Laplacian eigenvalues is one of the fundamental topics in graph theory and spectral integral variation. In this paper, we introduced the basic notions and theorems about majorization, threshold graphs, and Laplacian eigenvalues, and we surveyed some of the recent results in this area.

KEYWORDS : Laplacian matrix, Spectral integral variation, Laplacian integral.

1.INTRODUCTION

In this paper, we will provide an overview of parts of the theory of spectral graphs that are useful and have many important applications in computer science. Spectral graph theory is a mathematical theory where linear algebra and graph theory meet. The construction of spectral graph theory using the eigenvalues of a matrix M has been studied for about the past ten years.[8] Let M matrix and dimension $m \times n$ is an arrangement of mn entries in m rows and n columns. If M is an $m \times n$ matrix, the entry in the intersection of the k th row and l th column is donated by m_{kl}, and we write $M = [m_{kl}]$. An $m \times 1$ matrix is called a column vector or , $1 \times n$ matrix is called a row vector), An $n \times n$ matrix is called a square matrix (of order n). Let $M_{n\times n} = [m_{ij}]$ be a square matrix. If $m_{kl} = 0$ for $k \neq l$, then M is called a diagonal matrix A diagonal matrix with (main) diagonal entries $m_{11} = \mu_1, m_{22} = \mu_2, \dots, m_{nn} = \mu_n$ denoted by diag$(\mu_1, \mu_2, \dots, \mu_n)$; and if $\mu_i = 1$ for each i, M is called the identity matrix (of order n), and it is denoted by I_n or (if its order is obvious) by I. If $m_{ij} = 0$ for each $i > j$ (respectively, $i < j$), then M is called an upper triangular matrix or, lower triangular matrix. It is obvious that M is upper triangular iff M^T is lower triangular.

The Laplacian matrix is defined as follows Given a graph Γ , where $\Gamma = (V , \mathrm{E})$, V is the set of nodes/vertices and E is the set of edges that is mean if Γ be a graph with $V(\Gamma) = [n]$ and $E(\Gamma) = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m\}$. Let $L(\Gamma) = [l_{ij}]$ be the square matrix of order n defined as follows: If $i \neq j$, $l_{ij} = -1$ if the vertices i and j are adjacent and $l_{ij} = 0$ otherwise; if $i = j, l_{ii} = \rho_i$, where $\rho_i = deg(i)$. Then $L(\Gamma)$ is called the "Laplacian matrix (LM)" of Γ.[4]
And if Γ be a graph with $V(\Gamma) = [n]$, and $j \in V(\Gamma)$. Then j is called dominating if $jl \in E(\Gamma)$ for every $l \in [n]\backslash\{j\}$ ın addıtıon to ıf assume the eigenvalues of $L(\Gamma)$ are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$.Then for any$x \in \mathbb{R}$, $L + xJ$ has eigenvalues: $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1}$ and nx we can explain that since L is symmetric , it is orthogonally diagonalizable. That is, there is an orthogonal matrix Q s.t. each of the columns of Q is an eigenvector of L.
assume that the last column of Q is $\left[\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}\right]^T$. (By the definition of L, $L\bar{1} = 0 = 0\ \bar{1}$. Therefore, $\bar{1}$ is an eigenvector of L belonging to the eigenvalue 0. And since $||\bar{1}|| = \sqrt{n}$, $\frac{1}{\sqrt{n}}\bar{1}$ a unit eigenvector of L corresponding to the eigenvalue 0.) Then:

$$Q^T L Q = diag\ (\mu_1,\ \mu_2, \dots, \mu_{n-1},\ \mu_n = 0).$$

Now, let $Q = [c_1, c_2, \dots, c_n]$, where c_j is the j th column of Q. Then, by definition of an orthogonal matrix, $||c_i|| = 1$ for any $i = 1,\ 2, \dots, n$, and $c_i . c_j = 0$ for any $i \neq j\ \in \{1, 2, \dots, n\}$. In particular the vector $\bar{1}$ is orthogonal to each column of Q except for the last column ,then : $\therefore\ Q^T L Q = diag\ (0, 0, \dots, 0, n)$ by the usual matrix product.

$$\therefore\ Q^T(L + xJ)Q = Q^T L Q + x\ \ Q^T J\, Q$$
$$= diag\ (\ \mu_1, \mu_2, \dots, \mu_{n-1}, 0) + x\ diag\ (0, 0, \dots 0, n)$$

$$= \ diag\ (\mu_1,\ \mu_2, \dots, \mu_{n-1}\ , nx).$$

$\therefore$ The eigenvalues of $L\ +\ xJ$: $\mu_1 \geq \mu_2 \geq \cdots \geq\ \mu_{n-1}$ and nx.

2 SPECTRAL INTEGRAL VARIATION IN LAPLACIAN EIGENVALUES

One of the well-known conjectures about the Laplacian spectrum of a graph was asserted by Grone and Merris (1994), and this conjecture claimed that the conjugate of a degree sequence of a graph majorizes the Laplacian characteristic values of this graph. That conjecture is solved by Bai (2011). the spectral integral variation of the Laplacian matrix in a graph encodes valuable information about its connectivity and structure. Its eigenvalues, known as Laplacian eigenvalues, play a crucial role in various applications, including network analysis, computer science, and physics.

Definition 2.6 Let Γ be a graph and let $\mu_1, \mu_2, \dots, \mu_n$ be the eigenvalues of Γ. If $\mu_j \in \mathbb{Z}$ for each $j = 1, 2, \dots, n,$ then Γ is called a Laplacian integral (LI) graph.[7]

From the last theorem of the previous section, a TG is LI. But there are more, that is, the set of LI graphs properly contains the set of TGs. Now, we define a new type of graph, which is LI, and TG belongs to the set of this type of graphs.

Definition 2.7 Let Γ be a graph. Γ is said to be a ″cograph″ if it is obtained recursively by obeying the following principles:

a) K_1, i.e., a single vertex is a cograph.

b) If Ω is a cograph, then $\overline{\Omega}$ is a cograph.

c) If Ω and Λ are two cographs s.t. $V(\Omega) \cap V(\Lambda) = \emptyset$, then $\Omega + \Lambda$ is a graph.

Proposition 2.8 If Γ is a cograph, then it is LI.

Proof. We use the recursive definition of cographs and the following two facts:

1) Ω and Λ are LI $\Rightarrow$ $\Omega + \Lambda$ is LI:

Since $L(\Omega+\Lambda)=\begin{bmatrix} L(\Omega) & 0 \\ 0 & L(\Omega) \end{bmatrix}$, the set the of eigenvalues $L(\Omega+\Lambda)$ is the union of the set of eigenvalues of $L(\Omega)$ and $L(\Lambda)$, counting with multiply .

Now, since Ω and Λ are LI their eigenvalues are all integers.

∴ The eigenvalues of $L(\Omega+\Lambda)$ are integers.

∴ $L(\Omega+\Lambda)$ is LI .

2) Γ is LI ⇒ $\bar{\Gamma}$ is LI:

We know from Step 3 of the proof of the last theorem of the previous section that for any graph Ω, we have:

$$\mu^{*}_{n-i}(\Omega^c) = n - \mu_i\ (\Omega) \text{ for each } 1 \leq i < n, \text{ where } n = |V(\Omega)|\ .$$

And, for any graph Ω, $\mu_n(\Omega)$ = 0:

Step 1. Let $L(\Omega) = [\,n\,]$ and $E(\Omega) = \{e_1, e_2, \dots, e_m\}$. Assume that each edge of Ω is given a direction. Let $B(\Omega) = B = [b_{ij}]$ be the $n \times m$ matrix, whose rows (respectively, columns) are indexed by $V(\Omega)$ (respectively, $E(\Omega)$) s.t.

$$b_{ij} = \begin{cases} 0, & if \quad i \notin e_j = \{v,w\}; \\ 1, & if\ i \in e_j = \{v,w\} \text{ is the initial vertex of } e_j; \\ -1, & if\ i \in e_j = \{v,w\} \text{ is the final vertex of } e_j; \end{cases}$$

Then, $B(\Omega) = B$ is called the "incidence matrix (IM)” of Ω.

Step 2. Let Ω be a connected graph. Then rank $B(\Omega) = n - 1$:

Let $v = [v_1, v_2, \dots, v_n]^T \in \mathbb{R}^n$ be s.t. v^TB = $\bar{0}$. Thus, by definition of B, if $i{\sim}j$ $(i\,.j \in V(\Omega))$, then $v_i - v_j$ = 0, (respectively, $v_j - v_i$ = 0) if i (respectively, j) is the initial vertex of the edge $e = \{i\,,j\}$.

Therefore, if there is an $v_a v_b$ - path between the vertexes v_a and v_b, then $v_a = v_b$:

Let P = i_1 = v_a, i_2, …, i_{m-1}, i_m= v_b be a $v_a\ v_b - p\ a$th. Then $i_1 \sim i_2, i_2 \sim i_3, \dots, i_{m-1} \sim i_m$.

Therefore, $v_a = v_{i_1} = v_{i_2},\ v_{i_2} = v_{i_3}, \dots,\ v_{i_{m-1}} = v_{i_m} = v_b$. Thus, $v_a = v_b$.

∴ Since there is an ij − path in Ω for any two vertices $i, j \in V(\Omega)$ (because Ω is connected), all the components of v must be equal to each other.

∴ dim ($\mathcal{N}(B)$) ≤ 1. (dim(left $\mathcal{N}$)) = dim (right $\mathcal{N}$) = dim ($\mathcal{N}$).)

∴ rank $(B) \geq n-1$.

On the other hand, by definition of B, the sum of entries in each column (in fact, there are only two non-zero entries, one of them is 1 and the other is -1) is zero. Therefore, the sum of all the rows of B is $[0,0,\dots,0]$. Thus, the rows of B are linearly independent.

∴ rank $(B) \leq n-1$.

∴ rank $(B) = n-1$.

Step 3. If Ω has k components, then rank $(B\ (\Omega)) = n-k$:

Let $\Omega_1,\ \Omega_2, \dots, \Omega_k$ be the components of Ω. Then, after relabeling the elements of $V(\Omega)$ and $E(\Omega)$ if necessary, IM, of Ω is the following block diagonal matrix:

$$B(\Omega) = \begin{bmatrix} B\ (\Omega_1) & 0 & \dots & 0 \\ 0 & B\ (\Omega_2) & \dots & 0 \\ & 0 & & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B\ (\Omega_k) \end{bmatrix}$$

Now, for each $i = 1,2,\dots,k$, since Ω_i is connected, rank $B(\Omega_i) = |V\ (\Omega_i)|-1$ by the previous step.

∴ rank($B(\Omega)$) = rank($B(\Omega_1)$) + rank ($B(\Omega_2)$) + ... + rank ($B(\Omega_k)$)

= $(|V(\Omega_1)|-1) + (|V(\Omega_2)|-1) + \dots + (|V(\Omega_k)|-1)$

= $(|V(\Omega_1)| + |V(\Omega_2)| + \dots + |V(\Omega_k)|) - k$

= $n-k$.

Step 4. For any matrix M, (rank (MM^T)) = rank (M^TM) = rank M:

$\mathcal{N}(M^TM) = \mathcal{N}(M)$:

$(\subseteq)$: $x \in \mathcal{N}(M^TM) \Rightarrow M^TMx = 0 \Rightarrow x^T\ \ M^TM_x = 0 \Rightarrow (M_n)^T(M_n) = 0$

$\Rightarrow ||Mx|| = 0 \Rightarrow Mx = \bar{0} \Rightarrow x \in \mathcal{N}(M)$.

$(\supseteq)$ $x \in \mathcal{N}(M) \Rightarrow Mx = 0 \Rightarrow M^TMx = 0 \Rightarrow x \in \mathcal{N}(M^TM)$.

∴ $\mathcal{N}(M^TM) = \mathcal{N}(M)$.

∴ nullity (M^TM) = nullity (M).

∴ Since rank + nullity = The number of columns of M, a fixed number, rank (M^TM) = rank (M).

Step 5. $L(\Omega) = B(\Omega)B(\Omega)^T$:

Let $L(\Omega) = L = [l_{ij}]$, $B(\Omega) = B = [b_{ij}]$ and $B(\Omega)^T = B^T = [b_{ij}^T]$. Then:

$i = j$: $l_{ii} = \rho_i = deg\ (i)$ by definition of L. And$(BB^T)_{ii} = \sum_{k=l}^{n} b_{ik}\ b_{ki}^T = \sum_{k=1}^{n} b_{ik}\ b_{ik} = \sum_{k=1}^{n} (b_{ik})^2 = \sum_{k\sim i} (b_{ik})^2 = \sum_{k\sim i} (\pm 1)^2 = \sum_{k\sim i} 1 = \rho_i = deg\ (i)$.

$\therefore\ l_{ii} = (BB^T)_{ii}$.

$i \neq j$: $l_{ij} = \begin{cases} 0, & if\ \ i \nsim j; \\ -1, & if\ \ i \sim j. \end{cases}$ And $(BB^T)_{ij} = \sum_{k=1}^{n} b_{ik}\ b_{kj}^T = \sum_{k=1}^{n} b_{ik}\ b_{jk} = \sum_{\substack{i \in e_k = \{v,\ w\} \\ j \in e_k = \{v,\ w\}}} b_{ik} b_{jk}$,

because if $i \notin e_k = \{v, w\}$ or if $j \notin e_k = \{v, w\}$, then $b_{ik} = 0$ or $b_{jk} = 0$, respectively. Then, $b_{ik} b_{jk} = 0$.

Therefore, if $i \nsim j$, i.e., if there is no edge with end vertices i and j, then there is no non-zero term in the sum. Thus, $(BB^T)_{ij} = 0$. And if $i \sim j$, i.e., if there is an edge. $e = \{i, j\}$, then there is only one non-zero term in the sum which corresponds to the term, say $k = m$, that is, $e_m = \{i, j\}$. In this case, "$b_{ik} = 1$ and $b_{jk} = -1$" or "$b_{ik} = -1$ and $b_{jk} = 1$", and thus $(BB^T)_{ij} = b_{im} b_{jm} = -1$.

$\therefore l_{ij} = (BB^T)_{ij}$.

As a result, $L = BB^T$.

Step 6. $rank\left(L(\Omega)\right) = rank(B(\Omega)B(\Omega)^T)$ (Step 5)

$= rank\left(B(\Omega)\right)$ (Step 4)

$\leq n - 1$ (Step 3).

$\therefore$ nullity $(L(\Omega)) \geq 1$.

$\therefore$ There is a non-zero vector in the null space of $L(\Omega)$.

$\therefore$ 0 is the eigenvalue of $L(\Omega)$, that is, $\mu_n(\Omega) = 0$.

(Note that $L(\Omega) = B(\Omega)B(\Omega)^T \Rightarrow L(G)$ is (symmetric) and positive semi-definite matrix $\Rightarrow$ Each eigenvalue of $L(\Omega) \geq 0$.)

Finally: Γ is $LI \Rightarrow \mu_i(\Gamma) \in \mathbb{Z}$ for each $1 \leq i \leq n$.

$\Rightarrow$ Since $\mu_{n-i}(\Gamma^c) = n - \mu_i(\Gamma)$ for each $1 \leq i \leq n$, $\mu_{n-i}(\Gamma^c) \in \mathbb{Z}$ for each $1 \leq i \leq n$.

$\Rightarrow$ Since also $\mu_n(\Gamma^c) = 0 \in \mathbb{Z}$, $\mu_i(\Gamma^c) \in \mathbb{Z}$ for each $1 \leq i \leq n$.

$\Rightarrow \Gamma^c$ is LI.

Now, by using these two facts, we can prove the proposition:

First of all, K_1 is trivially LI. Let Ω and Λ be two cographs. Assume that both are LI. Then the cograph $\bar{\Omega}$ is LI by Fact (2) above, and the cograph $\Omega + \Lambda$ is LI by Fact (1) above.

∴ Recursively obtained each cograph is also LI.

As a result, if Γ is a cograph, then it is LI.

Proposition 2.9 If Γ is a TG, then it is a cograph.

Proof. Wlog, we can assume that Γ is connected, because if Γ is disconnected, then it has only one nontrivial component, and (if exists) the remaining components are all trivial, i.e., each remaining component comprises of a single vertex; thus, they are all cographs, and disjoint union cographs is a cograph.

Now, proof can be done by induction on $|V(\Gamma)| = n$:

$n = 1$: $\Gamma = K_1 \Rightarrow \Gamma$ is a cograph.

Assume that the proposition is true for all graphs with the number of vertices $\leq n - 1$.

Let Γ be a connected TG with $|V(\Gamma)| = n > 1$.

Since Γ is a nontrivial TG, it has a dominating vertex, say $v \in V(\Gamma)$. Let $\Omega = \Gamma \backslash \{v\}$.

Then, Ω is a TG with $|V(\Omega)| = n - 1$. Therefore, by the induction hypothesis, Ω is a cograph. Thus, $\bar{\Omega}$ is a cograph.

∴ $\bar{\Omega} + \{v\}$ is a cograph.

∴ $\overline{\bar{\Omega} + \{v\}} = \bar{\bar{\Omega}} \vee \{v\} = \Omega \vee \{v\} = \Gamma$ is a cograph.

As a result, the proposition is proved by induction.

Note 2.10 The converse of the previous proposition does not hold. That is, there are cographs that are not threshold. For example, C_4 is a cograph ($K_1$is a cograph, by part (a) of the definition of cograph. $4K_1 = K_1 + K_1 + K_1 + K_1$ is a cograph part (c) of the definition of cograph. Thus, $\overline{4K_1} = C_4$ is a cograph by part (b) of the definition of cograph); but C_4 is not a TG. (C_4 is connected, but does not have a dominating vertex.)

Now, Let Γ be a graph, and e be an edge not in Γ. Then, we want to understand the variation in the eigenvalues of $L(\Gamma)$ if we add e to Γ. To study this variation, we need a result from Linear Algebra about the consequence of the rank 1 perturbation process on the eigenvalues of A, where A is a square matrix of order n and $A^T = A$. (A square matrix $B = A + xy^T$, where $x, y \in \mathbb{R}^n$, is called a "rank 1 perturbation" of A.)

First, we study two lemmas which are needed to prove the mentioned result.

Lemma 2.11 Let M be a square matrix of order n, and $M^T = M$. Partition M as follows:

$$M = \begin{bmatrix} m_{11} & y^T \\ y & M(1|1) \end{bmatrix}.$$

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be the eigenvalues of M. Assume that the set of eigenvalues of $M\,(1|1)$ is a subset of the set of eigenvalues of M. Then y is the zero vector.

Proof. First, by the assumption, the eigenvalues of $M\,(1|1)$ are

$\alpha_1, \alpha_2, \dots, \alpha_{t-1}, \alpha_{t+1}, \dots, \alpha_n$ for some $t \in \{1, 2, \dots, n\}$.

Then we have:

$$tr(M) - tr\big(M(1|1)\big) = \alpha_t.$$

And since the trace of the square of a matrix A is equal to the sum of the squares of eigenvalues of A, we also have:

$$tr(M^2) - tr(M(1|1)^2) = \alpha_t^2.$$

On the other hand, from the statement of the lemma, we have:

$$tr(M) - tr\big(M(1|1)\big) = m_{11}.$$

And since $M^2 = \begin{bmatrix} m_{11}^2 + y^T y & m_{11}y^T + y^T M(1|1) \\ y m_{11} + M(1|1)y & yy^T + M(1|1)^2 \end{bmatrix}$, we have:

$$\begin{aligned} tr(M^2) &= m_{11}^2 + y^T y + tr(yy^T + M(1|1)^2) \\ &= m_{11}^2 + y^T y + tr(yy^T) + tr(M\,(1|1)^2) \\ &= m_{11}^2 + y^T y + tr(y^T y) + tr(M(1|1)^2) \\ &= m_{11}^2 + 2y^T y + tr(M(1|1)^2). \end{aligned}$$

$\therefore\ tr(M^2) - tr(M(1|1)^2) = m_{11}^2 + 2y^T y.$

$\therefore\ m_{11} = \alpha_t,\ m_{11}^2 + 2y^T y = \alpha_t^2.$

$\therefore\ y^T y = 0 \Rightarrow\ ||y||^2 = 0.$

$\therefore y = 0.$

Lemma 2.12 By the same notation of the previous lemma, let $0 \neq \delta \in \mathbb{R}$, and let

$$N = \begin{bmatrix} m_{11} + \delta & y^T \\ y & M\,(1|1) \end{bmatrix}$$

be a block matrix.

Assume that the eigenvalues of M (respectively, N) are $\alpha_1, \alpha_2, \dots, \alpha_n$ (respectively, $\alpha_1, \alpha_2, \dots, \alpha_{t-1}$, $\alpha_t + \delta, \alpha_{t+1}, \dots, \alpha_n$ for some $t \in \{1, 2, \dots, n\}$). Then y is the zero vector.

Proof. First of all, we have:

$c_M(\mu) = det(\mu I - M) = (\mu - \alpha_1)(\mu - \alpha_2) \dots (\mu - \alpha_n),$

$c_M(\mu) = det(\mu I - N) = (\mu - \alpha_1) \dots (\mu - \alpha_{t-1})(\mu - \alpha_t - \delta)(\mu - \alpha_{t+1}) \dots (\mu - \alpha_n).$

$\therefore \; c_N(\mu) = c_M(\mu) - \delta(\mu - \alpha_1) \dots (\mu - \alpha_{t-1})(\mu - \alpha_{t+1}) \dots (\mu - \alpha_n).$

On the other hand, from the statement of the lemma, we have:

$$c_N(\mu) = \begin{vmatrix} \mu - m_{11} - \delta & -y^T \\ -y & \mu I - M(1|1) \end{vmatrix} = \begin{vmatrix} \mu - m_{11} & -y^T \\ -y & \mu I - M(1|1) \end{vmatrix} + \begin{vmatrix} -\delta & 0^T \\ -y & \mu I - M(1|1) \end{vmatrix}$$

$$= det(\mu I - M) - \delta\, det(\mu I - M(1|1))$$

$$= c_N(\mu) - \delta\, det\big(\mu I - M(1|1)\big).$$

$\therefore \delta(\mu - \alpha_1) \dots (\mu - \alpha_{t-1}) (\mu - \alpha_{t+1}) \dots (\mu - \alpha_n) = \delta\, det(\mu I - M(1|1)).$

$\therefore$ Since $\delta \neq 0$ by assumption, we have:

$$det(\mu I - M(1|1)) = (\mu - \alpha_1) \dots (\mu - \alpha_{t-1}) (\mu - \alpha_{t+1}) \dots (\mu - \alpha_n).$$

$\therefore$ The eigenvalues of $M(1|1)$ comprises of $n - 1$ eigenvalues of M.

$\therefore$ By the previous lemma, y is the zero matrix.

Theorem 2.13 Let M and N be square matrices of order n s.t. $M^T = M$ and $N^T = N$, rank$(N) = 1$. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ (respectively, $0 \neq \delta, 0, \dots, 0$) be the eigenvalues of M (respectiely, N). Then matrix $M + N$ has the eigenvalues $\alpha_1, \dots, \alpha_{t-1}, \alpha_t\, \delta, \alpha_{t+1}, \dots, \alpha_n$ for some $t \in \{1, 2, \dots, n\}$ iff M and N commute.

Proof.

(⇐) Since M and N are commuting symmetric matrices, there is an orthogonal matrix P s.t.

$$PMP^T = diag(\alpha_1, \alpha_2, \dots, \alpha_n),\ PNP^T = diag(0, \dots, 0, \delta, 0, \dots 0),$$

where δ is the t th diagonal entry of the matrix PNP^T for some $t \in \{1, 2, \dots, n\}$.

Then we have:

$$P(M+N)P^T = PMP^T + PNP^T = diag(\alpha_1, \dots,\ \alpha_{t-1}, \alpha_t + \delta\ ,\ \alpha_{t+1}, \dots, \alpha_n),$$

for some $t \in \{1, 2, \dots, n\}$.

∴ The eigenvalues of $M+N$ are $\alpha_1, \dots,\ \alpha_{t-1}, \alpha_t + \delta\ ,\ \alpha_{t+1}, \dots, \alpha_n$ for some $t \in \{1, 2, \dots, n\}$.

(⇒) Wlog, we can suppose that $N = diag\ (\delta, 0, \dots, 0)$ (Since N is symmetric, there is an orthogonal matrix P s.t. $PNP^T = diag\ (\delta, 0, \dots, 0)$. And since (orthogonally) similar matrices have the same eigenvalues, both M and PMP^T, also both $M+N$ and $P(M+N)P^T$ have the same eigenvalues. Therefore, instead of studying with M and N, we can study with PMP^T and PNP^T. And thus we may suppose that $N = diag(\delta, 0, \dots, 0)$.)

Let $\quad M = \begin{bmatrix} m_{11} & y^T \\ y & M(1|1) \end{bmatrix}.$

Then $\quad M + N = \begin{bmatrix} m_{11} + \delta & y^T \\ y & M\ (1|1) \end{bmatrix}.$

Now, the eigenvalues of M(respectively, $M+N$) are $\alpha_1, \alpha_2, \dots, \alpha_n$ (respectively, $\alpha_1, \dots,\ \alpha_{t-1}, \alpha_t + \delta\ ,\ \alpha_{t+1}, \dots, \alpha_n$) for some $t \in \{1, 2, \dots, n\}$ by the hypothesis.

∴ By the previous lemma, y is the zero vector.

$\therefore M = \begin{bmatrix} m_{11} & 0^T \\ 0 & M(1|1) \end{bmatrix}.$

∴ Since $N = diag\ (\delta, 0, \dots, 0)$, M and N commute.

Now, we apply the previous theorem to Laplacian matrices.

Let Γ be a graph with $V(\Gamma) = [n]$, and suppose that $\{i, j\} \notin E(\Gamma)$. Let $\Omega = \Gamma + \{i, j\}$. Then:

$$L(\Omega) = L(\Gamma) + e_{ij} e_{ij}^T,$$

where $e_{ij} \in \mathbb{R}^n$ is the vector whose i th component is 1, j th component is -1, and all the remaining components are zero.

$\therefore$ By the related interlacing result, if $\alpha_1, \alpha_2, \ldots, \alpha_n = 0$ (respectively, $\beta_1, \beta_2, \ldots, \beta_n = 0$) are the eigenvalues of $L(\Gamma)$ (respectively, L(Ω)), then we have:

$$\beta_1 \geq \alpha_1 \geq \beta_2 \geq \alpha_2 \geq \cdots \geq \alpha_{n-1} \geq \beta_n \geq \alpha_n.$$

Also note that:

$$tr(L(\Omega)) = tr(L(\Gamma) + e_{ij}\, e_{ij}^T) = tr(L(\Gamma)) + tr(e_{ij}\, e_{ij}^T) = tr(L(\Gamma)) + 2. \qquad (*)$$

Assume that Γ is LI. Also assume in consideration of (*) above that one of the following holds:

a) Either $\beta_{j_m} = \alpha_{i_m}$ for $m = 1, 2, \ldots, n-1$, and $\beta_{j_n} = \alpha_{i_n} + 2$;

b) Or $\beta_{j_m} = \alpha_{i_m}$ for $m = 1, 2, \ldots, n-2$, and $\beta_{j_{n-1}} = \alpha_{i_{n-1}} + 1$.

Then, in consideration of (*) above, Ω is also LI.

Definition 2.14 If case (a) (respectively, (b)) occurs, then it is said that "spectral integral variation happens in 1 (respectively, 2) position(s)."

The following theorem characterizes case (a).

Theorem 2.15 Let Γ be a graph with $V(\Gamma) = [n]$ and $\{i, j\} \notin E(\Gamma)$. Let $\Omega = \Gamma + \{i, j\}$. Then, $n - 1$ eigenvalues of $L(\Gamma)$ and $L(\Omega)$ concur iff $N(i) = N(j)$.

Proof. First, we showed above that $(\Omega) = L(\Gamma) + e_{ij}\, e_{ij}^T$. By the previous theorem, $n-1$ eigenvalues of $L(\Gamma)$ and $L(\Omega)$ concur.

$\Leftrightarrow L(\Gamma)\ e_{ij}\, e_{ij}^T = e_{ij}\, e_{ij}^T\ L(\Gamma)$.

$\Leftrightarrow$ (By block multiplication) $(c_i - c_j)\, e_{ij}^T = e_{ij}\,(r_i - r_j)$, where c_k (respectively, r_k) is the k th column (respectively, k th row) of $L(\Gamma)$ for $k = i, j$.

$\Leftrightarrow$ (r_i^T - r_j^T) $e_{ij}^T = e_{ij}\,(r_i - r_j)$ (because $L(\Gamma)$ is a symmetric matrix).

$\Leftrightarrow (r_i - r_j)^T\, e_{ij}^T = e_{ij}\,(r_i - r_j)$.

$\Leftrightarrow (e_{ij}\,(r_i - r_j))^T = e_{ij}(r_i - r_j)$.

$\Leftrightarrow$ (By block multiplication) since

$e_{ij}\,(r_i - r_j) = \left[0, \dots, 0, (r_i - r_j)^T, 0, \dots, 0, (r_j - r_i)^T, 0, \dots, 0\right]^T,$

$r_i - r_j = \left[0, \dots, 0, l_{ii} - l_{ji}, 0, \dots, 0, l_{ji} - l_{ii}, 0, \dots, 0\right]$, where $L(\Gamma) = [l_{ji}]$.

$\Leftrightarrow (r_i - r_j)_k = l_{ik} - l_{jk} = 0$ for each $k \neq i, j$ and$(r_i - r_j)_i = l_{ii} - l_{ji}$, $(r_i - r_j)_j = l_{ji} - l_{ii}$

$\Leftrightarrow l_{ik} = l_{jk}$ for each $k \neq i, j$and $l_{ii} - l_{ji} = l_{ii} - l_{ji}$, $l_{ij} - l_{jj} = l_{ji} - l_{ii}$ (Note that since $\{i, j\} \notin E(\Gamma)$, $l_{ij} = l_{ji} = 0$.)

$\Leftrightarrow l_{ik} = l_{jk}$ for each $k \neq i, j$ and $l_{ii} = l_{jj}$.

$\Leftrightarrow$ The i th and j th rows of $L(G)$ are identical.

$\Leftrightarrow N(i) = N(j)$.

Corollary 2.16 Let Γ be a graph with $V(\Gamma) = [n]$ and $\{i, j\} \notin E(\Gamma)$, also assume that $N(i) = N(j)$. And let $\Omega = \Gamma + \{i, j\}$. Then Γ is LI iff Ω is LI.

Proof. First of all, since $N(i) = N(j), (n-1)$ eigenvalues of $L(\Gamma)$ and $L(\Omega)$ concur by the previous theorem. Therefore, Γ is LI iff these concurrent eigenvalues and the remaining eigenvalue of $L(\Gamma)$ are all integers iff (Since $tr(\Omega) = tr(\Gamma) + 2$,) these concurrent eigenvalues and the remaining eigenvalue of $L(\Omega)$ are all integers iff Ω is LI.

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