LAPLACIAN EIGENVALUES OF THRESHOLD GRAPHS AND MAJORIZATION

ABSTRACT

The topic of Laplacian eigenvalues of threshold graphs is one of the fundamental topics in graph theory. In this paper, we introduced the basic notions and theorems about majorization, threshold graphs, and Laplacian eigenvalues, and we surveyed some of the recent results in this area**.**

KEYWORDS : graph , threshold graph; Laplacian matrix; Majorization

1.INTRODUCTION

Graphs can be thought of as links between things. To emphasize a real problem, these things are linked by some relationship This paper is concerned with threshold graphs, introduced by Chvátal and Hammer [3] and Henderson and Zalestein in 1977. They are an important class of graphs because of their numerous applications [1].threshold graphs (TGs) as special graphs having beautiful structures and several important mathematical properties. It has a large impact in graph theory (GT) as well as in many applied areas, such as psychology, artificial intelligence, computer science, etc [2] . These graphs can also be used to control flow of information between processors, similar to how traffic lights are used in controlling the flow of traffic.

2. MAJORIZATION

Definition 2.1 *Let* $a = [a_1, a_2, \dots, a_n]^T \in \mathbb{R}^n$, and let $a_{[1]} \geq a_{[2]} \geq \dots \geq a_{[n]}$ be a nonincreasing sequence of the components of the vector a. That is:

$a_{[1]} = max\{a_1, a_2, \dots, a_n\}$, and $a_{[j]} = max(\{a_1, a_2, \dots, a_n\} \backslash \{a_{[1]}, a_{[2]}, \dots, a_{[j-1]}\})$ for each $j = 2, 3, \dots, n$. Let $b = [b_1, b_2, \dots, b_n]^T \in \mathbb{R}^n$. If we have:

i) $\sum_{j=1}^{k} a_{[j]} \leq \sum_{j=1}^{k} b_{[j]}$, for each $k = 1, 2, \dots, n-1$,

ii) $\sum_{j=1}^{n} a_{[j]} \leq \sum_{j=1}^{n} b_{[j]}$,

then we say that a is "majorized" by b, or b "majorizes" a, and it is denoted by $a < b$. If $a < b$, we frequently say that $a_1, a_2, \dots, a_n$ are majorized by $b_1, b_2, \dots, b_n$. If c and d are $1 \times n$ real vectors, and if $c^T < d^T$, then we say that c is majorized by d, and it is also denoted by $c < d$.

Note 2.1 Let $a = [a_1, a_2, \dots, a_n]^T \in \mathbb{R}^n$, and let $\bar{a} = \frac{a_1 + a_2 + \dots + a_n}{n}$ be the arithmetic mean of the components of the vector a. Also let b = $[\bar{a}, \bar{a}, \dots, \bar{a}]^T$. Then $< a$:

First of all, $\sum_{j=1}^{n} b_{[j]} = \sum_{j=1}^{n} \bar{a} = n\ \bar{a} = a_{1+} a_{2+} \dots {}_{+}a_n = \sum_{j=1}^{n} a_{[j]}$.

∴ The condition (ii) in the previous definition is satisfied.

Secondly, $\bar{a} \leq a_{[1]}$: Suppose not. That is, $\bar{a} > a_{[1]}$.

Then, since $a_{[1]} \geq a_j$ for each $j = 1, \dots, n$, $\bar{a} > a_{[j]}$ for each $j = 1, \dots, n$.

$\Rightarrow\ n\bar{a} > a_{[1]} + a_{[2]} + \dots + a_{[n]} = a_{1+} a_{2+} \dots {}_{+}a_n$, a contradiction.

Now, we can show that the condition (i) of the previous definition is also satisfied:

Suppose to the contrary that there is some $k \in \{2, 3, \dots, n-1\}$ s.t. $\sum_{j=1}^{k} b_{[j]} > \sum_{j=1}^{k} a_{[j]}$. Let m be the smallest such k. That is, $\sum_{j=1}^{m-1} b_{[j]} \leq \sum_{j=1}^{m-1} a_{[j]}$, but $\sum_{j=1}^{m} b_{[j]} > \sum_{j=1}^{m} a_{[j]}$.

$\Rightarrow$ In particular, $b_{[m]} = \bar{a} > a_{[m]}$.

$\Rightarrow$Since $a_{[m]} \geq a_{[j]}$ for each $j = m, m+1, \dots, n$, $\bar{a} > a_{[j]}$ for each $j = m, m+1, \dots, n$.

$\Rightarrow$ Since $\sum_{j=1}^{m} b_{[j]} > \sum_{j=1}^{m} a_{[j]}$, $\sum_{j=1}^{m} b_{[j]} + (n-m)\,\bar{a} > \sum_{j=1}^{m} a_{[j]} + \sum_{j=m+1}^{n} a_{[j]}$.

$\Rightarrow \sum_{j=1}^{n} b_{[j]} > \sum_{j=1}^{n} a_{[j]} = a_{[1]} + \dots + a_{[n]} = a_1 + \dots + a_n$

$\Rightarrow\ n\bar{a} > a_{1+} a_{2+} \dots {}_{+}a_n$, a contradiction.

∴ The condition (i) of the previous definition must also be satisfied. As a result,

$[\bar{a}, \bar{a}, \dots, \bar{a}]^T < [a_{1+} a_{2+} \dots {}_{+}a_n]^T$.

3.LAPLACIAN EIGENVALUES

Let Γ be a graph with $V(\Gamma) = [n]$ and $E(\Gamma) = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m\}$. Let $L(\Gamma) = [l_{ij}]$ be the square matrix of order n defined as follows: If $i \neq j$, $l_{ij} = -1$ if the vertices i and j are adjacent and $l_{ij} = 0$ otherwise; if $i = j, l_{ii} = \rho_i$, where $\rho_i = deg(i)$. Then $L(\Gamma)$ is called the "Laplacian matrix (LM)" of Γ

Suppose that $\zeta_1, \zeta_2, \dots, \zeta_n$ are the eigenvalues of $L(\Gamma)$, and $\rho_1, \rho_2, \dots, \rho_n$ are the degrees of vertices. Then we have: $\rho = [\rho_1, \rho_2, \dots, \rho_n] < \zeta = [\zeta_1, \zeta_2, \dots, \zeta_n]$. et $s_1, s_2, \dots, s_n \in \mathbb{Z}$, and assume that $s_k > s_l$ for some $1 \leq k \neq l \leq n$. Then, define: $s'_k = s_k - 1$, $s'_l = s_l + 1$, and $s'_m = s_m$ for all $m \neq k, l$.Then $s'_1, s'_2, \dots, s'_n$ are said to be gotten from $s_1, s_2, \dots, s_n$ by a "transfer (from k to l)". Let $s, t \in \mathbb{Z}^n$. If the components of t are gotten from the components of s by a transfer, then t is said to be gotten from s by a "transfer".[4,6]

Corollary 3.1 (A corollary of the previous theorem)
Let Γ be a graph with $V(\Gamma) = [n]$, and let $deg(j) = \rho_j$ for each $j = 1, 2, \dots, n$. Then, $\rho_1^*, \rho_2^*, \dots, \rho_n^*$ majorizes $\rho_1, \rho_2, \dots, \rho_n$.

Proof. $A(\Gamma)$ is a (0, 1)-matrix s.t. both i th row sum and i th column sum of $A(\Gamma)$ is p_i for each i = 1, 2, ..., n.
∴ By the previous theorem, $\rho_1^*, \rho_2^*, \dots, \rho_n^*$ majorizes $\rho_1, \rho_2, \dots, \rho_n$. ■

4.THRESHOLD GRAPHS

Definition 4.1 Let Γ be a graph with $V(\Gamma) = [n]$, and let $j \in V(\Gamma)$. Then j is called "dominating" if $jl \in E(\Gamma)$ for every $l \in [n] \backslash \{j\}$.

Definition 4.2 Let Γ be a graph with $V(\Gamma) = [n]$. Assume that Γ is constructed recursively as follows:
Begin with K_1.
Then apply the following (i) or (ii) process finitely many times in any order: Let Γ be denote the present graph at each step, and let K_1 denote a new vertex not in $V(\Gamma)$.

i) $\Gamma + K_1$.

ii) $\Gamma \vee K_1$.

Then Γ is called a "threshold graph (TG)".

Now, let Γ be a given graph. How can we understand whether Γ is a TG or not? Is there a recursive process or an algorithmic procedure to determine whether Γ is a TG or not? The answer is affirmative:

Case 1. Γ is connected.

The first necessary condition for Γ to be a TG is to have a dominant $v \in V(\Gamma)$. Then the second necessary condition is that $\Gamma \backslash \{v\}$ has only one nontrivial component, say $\Omega \subseteq \Gamma \backslash \{v\}$ (and, there may be some trivial components). Moreover, Γ is a TG iff Ω is a TG.

Case 2. Γ is disconnected.

The first necessary condition for Γ to be a TG graph is that Γ has only one nontrivial component, say $\Omega \subseteq \Gamma$ (and, there may be some other trivial components). Moreover, Γ is a TG iff Ω is a TG.

Now, to prove the main result of this section, we first state and prove two lemmas:

Lemma 4.3 Let Γ be a graph with $V(\Gamma) = [n]$. Assume that the eigenvalues of $L(\Gamma)$ are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$.Then for any$x \in \mathbb{R}$, $L + xJ$ has eigenvalues: $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1}$ and nx.

Proof. First of all, since L is symmetric, it is orthogonally diagonalizable. That is, there is an orthogonal matrix Q s.t. each of the columns of Q is an eigenvector of L.

Wlog, assume that the last column of Q is $\left[\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right]^T$. (By the definition of L, $L\bar{1} = 0 = 0\,\bar{1}$. Therefore, $\bar{1}$ is an eigenvector of L belonging to the eigenvalue 0. And since $||\bar{1}|| = \sqrt{n}$, $\frac{1}{\sqrt{n}}\bar{1}$ a unit eigenvector of L corresponding to the eigenvalue 0.) Then:

$$Q^T L Q = diag\,(\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n = 0).$$

Now, let $Q = [c_1, c_2, \dots, c_n]$, where c_j is the j th column of Q. Then, by definition of an orthogonal matrix, $||c_i|| = 1$ for any $i = 1,\ 2, \dots, n$, and $c_i . c_j = 0$ for any $i \neq j\ \in \{1, 2, \dots, n\}$. In particular the vector $\bar{1}$ is orthogonal to each column of Q except for the last column. Therefore, by the usual matrix product, we have:

$$JQ = \begin{bmatrix} 0 & 0 & \dots & 0 & \sqrt{n} \\ 0 & 0 & \dots & 0 & \sqrt{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \sqrt{n} \end{bmatrix}$$

$\therefore\ Q^T LQ = diag\ (0, 0, \dots, 0, n)$ by the usual matrix product.

$$\therefore\ Q^T(L + xJ)Q = Q^T LQ + x\ \ Q^T J\, Q$$
$$= diag\ (\mu_1, \mu_2, \dots, \mu_{n-1}, 0) + x\ diag\ (0, 0, \dots 0, n)$$
$$= diag\ (\mu_1,\ \mu_2, \dots, \mu_{n-1}, nx).$$

$\therefore$ The eigenvalues of $L + xJ$: $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1}$ and nx. ■

Lemma 4.4 Let Γ be a graph with $V(\Gamma) = [n]$. Let $deg\,(j) = \rho_j$, for each $j = 1, 2, \dots, n$, and assume that $n - 1 = \rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$. Let $\Omega = \Gamma \backslash \{1\}$. Then one of the eigenvalues of $L(\Gamma)$ is n. Moreover, assume that the eigenvalues of $L(\Gamma)$ are $\mu_2,\ \mu_3, \dots, \mu_{n-1}, n, 0$. Then, $\mu_2 - 1,\ \mu_3 - 1, \dots, \mu_{n-1} - 1$, 0 are the eigenvalues of $L(\Omega)$.

Proof. First, by definition of Laplacian, $L(\Gamma) = \begin{bmatrix} n-1 & -\bar{1}^T \\ -\bar{1} & L(\Omega) + I_{n-1} \end{bmatrix}$, where $\bar{1} = [1, 1, \dots, 1]^T \in \mathbb{R}^{n-1}$. (Since $\Omega = \Gamma \backslash \{1\}$, and since $deg_\Gamma\ (1) = n-1$, $deg_\Omega\,(j)\ deg_\Gamma\,(j) - 1$ for each $j = 2, 3, .., n$. Therefore, we add I_{n-1} to the block matrix corresponding to $L(\Omega)$.)

$\therefore L(\Gamma) + J_n = \begin{bmatrix} n & \bar{0}^T \\ \bar{0} & L(\Omega) + I_{n-1} + J_{n-1} \end{bmatrix}$, where J_n is the $n \times n$ square matrix with all entries equal to 1, and $\bar{0} = [0, 0, \dots, 0]^T \in \mathbb{R}^{n-1}$.

Now, by the previous lemma, since the eigenvalues of $L(\Gamma)$ are $\mu_2, \mu_3, \dots, \mu_{n-1}, n, 0$, the eigenvalues of $L(\Gamma) + J_n$ are $\mu_2, \mu_3, \dots, \mu_{n-1}$ and n with multiplicity 2.

$\therefore$ Since $L(\Gamma) + J_n = \begin{bmatrix} n & \bar{0}^T \\ \bar{0} & L(\Omega) + I_{n-1} + J_{n-1} \end{bmatrix}$, the eigenvalues of $L(\Omega) + I_{n-1} + J_{n-1}$ are $\mu_2, \mu_3, \dots, \mu_{n-1}, n$.

$\therefore$ The eigenvalues of $L(\Omega) + J_{n-1}$ are $\mu_2 - 1, \mu_3 - 1, \dots, \mu_{n-1} - 1, n - 1$.

($det\,(\mu I_{n-1} - (L(\Omega) + J_{n-1} + I_{n-1})) = \det((\mu - 1)I_{n-1} - (L(\Omega) + J_{n-1}))$.

$\therefore \mu$ is an eigenvalue of $L(\Omega) + J_{n-1} + I_{n-1}$ iff $\mu - 1$ is an eigenvalue of $L(\Omega) + J_{n-1}$.)

$\therefore$ By the previous lemma again, the eigenvalues of $L(\Omega)$ are $\mu_2 - 1, \dots, \mu_{n-1} - 1, 0$. ■

Theorem 4.5 Let Γ be a graph with $V(\Gamma) = [n]$. Let $deg(j) = \rho_j$, for each $j = 1, 2, \dots, n$. Then the eigenvalues of $L(\Gamma)$ are $\rho_1^*, \rho_2^*, \dots, \rho_n^*$ iff Γ is a TG.

Proof.

($\Leftarrow$) Proof can be done by induction on the number of vertices.

$n = 1$: First of all, since we study only simple graphs, $L(\Gamma) = [0]$. And the only eigenvalue of the zero matrix is 0. Also, $deg(1) = 0 = \rho_1$ and $\rho_1^* = |\{\rho_j : \rho_j \geq 1\}| = 0$, because there is only are ρ_j which is ρ_1 and $\rho_1 = 0$. As a result, the only eigenvalue of $L(\Gamma)$ is $0 = \rho_1^*$.

Suppose that the result holds for all TGs with the number of vertices $\leq n - 1$.

Now, let Γ be a TG with $|V(\Gamma)| = n$.

First of all, it is enough to demonstrate the conclusion for a connected case, because each vertex of degree zero appends a 0 both to the sequence of degrees of Γ and to the list of eigenvalues of $L(\Gamma)$, simply because $0^* = 0$.

$\therefore$ Wlog, suppose that Γ is connected.

Now, by definition of being a TG, Γ has a vertex with degree $n - 1$, say $deg(1) = n - 1$.

Let $\Omega = \Gamma \backslash \{1\}$.

Let $\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n = 0$ be the eigenvalues of $L(\Gamma)$. Then, by the previous lemma, one of $\mu_j = n$ for some $j \in \{1, 2, \dots, n - 1\}$, say $\mu_1 = n$.

And by the previous lemma, $\mu_2 - 1, \mu_3 - 1, \dots, \mu_{n-1} - 1$, 0 are the eigenvalues of $L(\Omega)$.

As mentioned above, if k vertices each of which has degree 0, are added to a graph, then k 0s are appended both to the sequence of degrees of the graph and the list of eigenvalues of the Laplacian of the graph.

∴ If the statement of the theorem holds for a graph, then it also holds if we add some vertices, each of which has degree 0, to that graph. As a result, since Ω has only one nontrivial component, which is threshold, and there may be some other trivial components; the eigenvalues of $L(\Omega)$, namely, $\mu_2 - 1, \mu_3 - 1, \dots, \mu_{n-1} - 1, 0$, satisfy the statement of the theorem by the induction hypothesis. That is, since ρ_2-1, ρ_3-1,…, $\rho_n - 1$ is the sequence of degrees of the vertices in Ω, $(\rho_j - 1)^* = \mu_j - 1$ for each $j = 2, 3, \dots, n-1, (\rho_n - 1)^* = 0$. On the other hand, since $\rho_1 = n - 1$ (that is, each vertex is adjacent to the vertex 1, and thus $\rho_j \geq 1$ for each $j = 1, 2, \dots, n$), $\rho_1^* = |\{\rho_j : \rho_j \geq 1\}| = n$. Therefore, since $\mu_1 = n$, $\rho_1^* = \mu_1$. Finally, for clearness and simplicity of the notation, say $\tau_{j-1} = \rho_j - 1$ for each $j = 2, 3, \dots, n-1$. Then we have:

$$\begin{aligned}(\rho_j - 1)^* = \tau_{j-1}^* &= |\{\tau_l : \tau_l \geq j-1\}| \\ &= |\{\rho_{l+1} - 1 : \rho_{l+1} - 1 \geq j-1\}| \\ &= |\{\rho_{l+1} - 1 : \rho_{l+1} \geq j\}| \\ &= |\{\rho_k - 1 :\ \rho_k \geq j\}| \\ &= |\{\rho_k :\ \rho_k \geq j\}|\end{aligned}$$

The last equation is true because the matter is not the set, the matter is only the number of elements in the set.

∴ $(\rho_j - 1)^* = |\{\rho_k : \rho_k \geq j\}|$ for the sequence $\rho_2 - 1, \rho_3 - 1, \dots, \rho_{n-1} - 1, 0$, for each $j = 2, 3, \dots, n$.

∴ For the sequence $n - 1 = \rho_1, \rho_2, \rho_3, \dots, \rho_{n-1}, \rho_n = 0$, we have:

$$\rho_j^* = |\{\rho_l :\ \rho_l \geq j\}| = (\rho_j - 1)^* + 1, \text{for } j = 2, 3, \dots, n,$$

because $\rho_1$is the largest possible number in the sequence $\rho_1, \rho_2, \dots, \rho_n = 0$, and thus $\rho_l \geq j$ implies $\rho_1 \geq j$. (∴ We add +1 to $(\rho_j - 1)^*$ to find ρ_j^*.)

∴ $\rho_j^* = (\rho_j - 1)^* + 1 = (\mu_j - 1) + 1 = \mu_j$ for each $j = 2, 3, \dots, n$.

And we know that $\rho_1^* = n = \mu_1$.

As a result, $\rho_j^* = \mu_j$ for each $j = 1, 2, \dots, n$.

(⇒)(This part of the proof is taken from Merris (1994).)

The proof can be done easily by induction on the number of vertices n of Γ, and by using the recursive definition of a TG.

Step 1. First of all, if $\Gamma = K_n^c$, then Γ is a TG by definition.

If $\Gamma \neq K_n^c$, then there exists, say $m > 1$ vertices of Γ with vertex degree > 0. Let Ω be the subgraph of Γ induced by these m vertices. Then:

$$L(\Gamma) = \begin{bmatrix} L(\Omega) & 0_{m\times(n-m)} \\ 0_{(n-m)\times m} & 0_{n-m} \end{bmatrix}.$$

$\therefore$ For each $i = 1, 2, \ldots, m$ for the following first two equations and for each $i = 1, 2, \ldots, m-1$ for the following last equation, we have:

$\mu_i(\Gamma) = \mu_i(\Omega), \rho_i(\Gamma) = \rho_i(\Omega), \rho_i^*(\Gamma) = \rho_i^*(\Omega)$,

where $\mu_i(\Gamma), \rho_i(\Gamma)$, and $\rho_i^*(\Gamma)$ are the *i* th largest eigenvalue, the *i* th largest degree (in the degree sequence), and the *i* th conjugate element (in the conjugate sequence) of Γ. Similarly, $\mu_i(\Omega), \rho_i(\Omega)$, and $\rho_i^*(\Omega)$ denote the same notions for Ω.

Step 2. Now, $\rho_1^*(\Omega) = m. (\rho_1^*(\Omega) = |\{\rho_i(\Omega) | \rho_i(\Omega) \geq 1\}| = m$, because every vertex of Ω has a degree > 0, i.e., has a degree $\geq 1.)$ Therefore, since $\mu_1(\Gamma) = \rho_1^*(\Gamma)$ by hypothesis and $\mu_1(\Omega) = \mu_1(\Gamma)$, $\mu_1(\Omega) = \rho_1^*(\Omega) = \rho_1^*(\Gamma) = m$.

Step 3. Furthermore, for any graph Λ,

$$\mu_i(\Lambda) = n - \mu_{n-i}(\Lambda^c), 1 \leq i < n\ (***):$$

$L(\Lambda) + L(\Lambda^c) = nI_n - J_n$ by the definition of LM.

Therefore, since $L(\Lambda)$commutes with itself, with I_n, and with J_n, and since $L(\Lambda^c) = nI_n - J_n - L(\Lambda), L(\Lambda)$ commutes with $L(\Lambda^c)$. Then it is a well-known fact from Linear Algebra that the matrices $L(\Lambda)$ and $L(\Lambda^c)$ are simultaneously triangularizable. That is, there exists an invertible matrix, P s.t. $P^{-1}L(\Lambda)\,P$ and $P^{-1}L(\Lambda^c)\,P$ are both triangular matrices. Now, the eigenvalues of a triangular matrix are the main diagonal entries of that matrix. Therefore, we have:

$P^{-1}(L(\Lambda) + L(\Lambda^c))\,P = P^{-1}\,(nI_n - J_n)\,P$

$\Rightarrow P^{-1}\,L\,(\Lambda)\,P + P^{-1}\,L\,(\Lambda^c)\,P = nI_n - P^{-1}J_n\,P$.

On the other hand, the eigenvalues of $nI_n - P^{-1}J_n\,P$ are n with multiplicity $n-1$ and 0 with multiplicity 1. (First, since J_n and $P^{-1}J_n\,P$ are similar matrices, they have the same eigenvalues.

J_n is a symmetric matrix and rank(J_n) = 1. Therefore, there is only one nonzero eigenvalue of J_n. Thus, since the sum of all the eigenvalues is equal to the trace, and since ($n-1$) of the eigenvalues of J_n is 0, the unique nonzero eigenvalue of J_n must be $n = tr(J_n)$. Therefore, the eigenvalues of J_n are 0 with multiplicity $n-1$ and n with multiplicity 1. As a result, the eigenvalues of $nI_n - J_n$ are n with multiplicity $n-1$ and $n-n = 0$with multiplicity 1: $det(I_n - (nI_n - J_n)) = det((x-n)I_n - (-J_n)$. Now, $det(xI_n - (-J_n)) = 0 \Leftrightarrow x = 0$ with multiplicity $n-1$ and $x = -n$ with multiplicity 1. Therefore, $det((x-n)I_n - (-J_n)) = 0 \Leftrightarrow (x-n) = 0$ with multiplicity $n-1$ and $(x-n) = -n$ with multiplicity 1 $\Leftrightarrow$ $x = n$ with multiplicity $n-1$ and $x = 0$ with multiplicity 1.)

$\therefore$ The eigenvalues of $P^{-1}\ L(\Lambda)\,P+\ P^{-1}\ L(\Lambda^c)\,P$ are n with multiplicity $n-1$ and 0 with multiplicity 1.

$\therefore$ Since the eigenvalues of $P^{-1}\ L(\Gamma)\,P+\ P^{-1}\ L(\Gamma^c)\,P$, which are the same as the eigenvalues of $L(\Gamma)+L(\Gamma^c)$, are the main diagonal entries, we have:

$$\mu_i(\Gamma) + \mu_{n-i}(\Gamma^c) = n \quad \text{for each} \quad 1 \le i < n. \qquad (*)$$

(Since $(P^{-1}L(\Gamma)\,P)_{ii} + (P^{-1}L(\Gamma^c)\,P)_{ii} = n$ for each $i = 1, 2, \ldots, n-1$, the constant n; if one of them is large, then the other must be small. That is: $\mu_1(\Gamma) + \mu_{n-1}(\Gamma^c) = n, \mu_2(\Gamma) + \mu_{n-2}(\Gamma^c) = n, \ldots, \mu_{n-1}(\Gamma) + \mu_1(\Gamma^c) = 0$.)

In addition, we know that the remaining eigenvalue of $L(\Gamma)$ is 0 with multiplicity 1 and $\bar{1}$ is one of the corresponding eigenvectors.

Step 4. Now, if we apply (*) to Ω, then we get:

$\mu_i(\Omega) + \mu_{m-i}(\Omega^c) = m$ for each $1 \le i < m$.

$\therefore$ Since $\mu_1(\Omega^c) = m$, $\mu_{p-1}(\Omega) = 0$.

$\therefore$ Ω^c is a disconnected graph.

$\therefore$ Since a graph and its complement can not be both disconnected, Ω must be connected.

$\therefore$ $\mu_{m-1}(\Omega) \neq 0$.

$\therefore$ Since $\mu_{m-1}(\Omega) = \rho^*_{m-1}(\Omega)$ by the hypothesis, $\rho^*_{m-1}(\Omega) \neq 0$.

$\therefore$ Since $\rho^*_{m-1}(\Omega) = |\{\rho_i(\Omega) : \rho_i(\Omega) \ge m-1\}|$, and since $\Delta(\Omega) \le m-1$ (because Ω contain m vertices), $\Delta(\Omega) = m-1$.

$\therefore$ $\rho_1(\Omega) = m-1$.

Step 5. Now, suppose that there are s vertices of degree $m-1$ in Ω.

Then for a uniquely determined subgraph Λ of Ω, we have :

$\Omega = \Lambda \, V K_s$.

Step 6. $\therefore\ \mu_{s+i}\,(\Omega) = \mu_i\,(\Lambda) + s,\ 1 \le i < t$, where $t = m - s$ is $|V(\Lambda)|$:

First of all, for any two graphs φ and ψ we have $(\varphi\ V\ \psi) = (\varphi^c + \psi^c)^c$:

$V(\varphi\ V\ \psi) = V(\varphi) \cup V(\psi) = V((\varphi^c + \psi^c)^c)$.

$e = \{\alpha, \beta\} \in E\,((\varphi^c + \psi^c)^c) \Leftrightarrow e = \{\alpha, \beta\} \notin E\,(\varphi^c + \psi^c)$

$\Leftrightarrow\ e \notin E(\varphi^c)$ or $e \notin E\,(\psi^c)$ or "one of the end vertices of e, say α, is in $V(\varphi^c) = V(\varphi)$ and β is in $V(\psi^c) = V(\psi)$".

$\Leftrightarrow e \in E(\varphi)$ or $e \in E(\psi)$ or "$\alpha \in V(\varphi)$ and $\beta \in V(\psi)$ ".

$\Leftrightarrow e\ \in E(\varphi\ V\ \psi)$

$\therefore (\varphi\ V\ \psi) = (\varphi^c + \psi^c)^c$.

Now, by using the equation given in Step 3, we can prove the required equation:

$\mu_{s+i}(\Omega) = \mu_{s+i}\,(\Lambda V K_s) = \mu_{s+i}((\Lambda^c + K_s^c)^c)$

$= m - \mu_{m-(s+i)}(\Lambda^c + K_s^c)$ ($\Omega = (\Lambda^c + K_s^c)^c$ has m vertices.)

$= m - \mu_{m-(s-i)}\,(\Lambda^c)$ (K_s^c is s isolated vertices. Thus, the corresponding block of K_s^c in $L(\Omega)$ is the $s \times s$ zero matrix.)

$= m - ((m - s) - \mu_{(m-s)\,-\,(m-s-i)}(\Lambda))$ (Λ has $m - s$ vertices)

$= s + \mu_i(\Lambda)$.

$\therefore \mu_{s+i}\,(\Omega) = \mu_i(\Lambda) + s$, where $1 \le i < t = m - s = |V\,(\Lambda)|$.

(Note that $1 \le i < (m - s) \Rightarrow -(m - s) < -i \le -1$

$\Rightarrow 0 < m - s - i \le m - s - 1$

$\Rightarrow 1 \le m - s - i \le m - s - 1 = t - 1 \le m - 1 < m$,

i.e, $1 \le m - s - i < m$. Therefore, we can apply $(***)$ in the first case above. Similarly, $1 \le m - s - i < m - s$, because $1 \le i < m - s$. Therefore we can apply $(***)$ in the second case above.)

Step 7. $\rho^*_{s+i}(\Omega) = \rho^*_i(\Lambda) + s, 1 \le i < t$:

$\rho^*_{s+i}\,(\Omega) = |\{\rho_j(\Omega): \rho_j(\Omega) \ge s + i\}| = |\{\rho_j((\Lambda V K_s): \rho_j((\Lambda V K_s) \ge s + i\}|$

$= |\{\rho_j(\Lambda V K_s) \colon \rho_j(\Lambda) + s \geq s + i\}| + s$

(Note that $v \in V(\Lambda V K_s) \Rightarrow v \in V(\Lambda)\ or\ v \in V(K_s)$.

$v \in V(\Lambda) \Rightarrow \rho_{\Lambda V K_s}(v) = \rho(v)_\Lambda + s$, because each vertex of Λ is adjacent to every vertex of K_s.

$v \in V(K_s) \Rightarrow \rho_{\Lambda V K_s}(v) = \rho_{K_s}(v) + |V(\Lambda)|$, because each vertex of K_s is adjacent to every vertex of Λ.

$\therefore v \in V(K_s) \Rightarrow \rho_{\Lambda V K_s}(v) = (s - 1) + (m - s) = m - 1 \geq s + i$ for each $1 \leq i < t$, because $1 \leq i < t = m - s \Rightarrow s + 1 \leq s + i < (m - s)\ ts = m \Rightarrow s + i < m - 1$.

Also, note that $\left|\{\rho_j(\Lambda V K_s) \colon \rho_j(\Lambda) \geq i\}\right| = \left|\{\rho_j(\Lambda) \colon \rho_j(\Lambda) \geq i\}\right|$, because only the order of the set does matter. In both cases, we count j's for which $\rho_j(\Lambda) \geq i$.)

Step 8. $\therefore$ Since $\rho^*_{s+i}(\Omega) = \rho^*_{s+i}(\Gamma) = \mu_{s+i}(\Gamma) = \mu_{s+i}(\Omega)$

(the second equation holds by hypothesis, and the other two equations hold by Step 1),

both $\mu_{s+i}(\Omega) = \mu_i(\Lambda) + s$ (Step 6) and $\rho^*_{s+i}(\Omega) = \rho^*_i(\Lambda) + s$ (Step 7) imply that $\rho^*_i(\Lambda) = \mu_i(\Lambda) + s$.

$\therefore \rho^*_i(\Lambda) = \mu_i(\Lambda)$ for each $1 \leq i < t = |V(\Lambda)|$.

(Note that $1 \leq i < t = m - s \Rightarrow s + 1 \leq s + i < m$. Therefore, the equations proved in Step 1 can be applied.)

Step 9. Since $\rho^*_i(\Lambda) = \mu_i(\Lambda)$ for each $1 \leq i < t = |V(\Lambda)|$, and since $t = |V(\Lambda)| = m - s < m \leq n$, $t < n$.

$\therefore$ Since Λ satisfies the hypothesis of the theorem, Λ is a TG by induction assumption.

$\therefore$ Since $\Omega = \Lambda V K_s$, Ω is a TG by the recursive definition of TGs.

$\therefore$ Since $\Gamma = \Omega + K_{n-m}$, Γ is a TG again by the recursive definition of TGs.

(Note that we can not apply induction assumption to the graph Ω, because $|V(\Omega)| = m = n = V(\Gamma)$ is possible.) ■

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