Global well-posedness and polynomial decay for a nonlinear viscoelastic equation with variable density and memory

Abstract

This paper investigates a nonlinear viscoelastic equation with variable density and memory. We study the global well-posedness and show the polynomial decay results with more general and weaker assumptions (compared with the previous studies) on the memory kernel.

Keywords: Viscoelastic equation; Variable density; Memory kernel; Global weak solutions; Polynomial decay.

MSC: 35B35; 35G05; 35G16; 35L30.

1 Introduction

In this paper, we consider the following viscoelastic equation with variable density and memory

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \mu\Delta u_{tt} + \displaystyle\int_0^t g(t-s)\Delta u(s)ds = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u|_{t=0} = u_0, u_t|_{t=0} = u_1 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\mu > 0$ and $\rho \geq 0$ are two constants will be specific later, Ω is a bounded domain of $\mathbb{R}^n$ ($n \geq 1$ is an integer) with a smooth boundary $\partial\Omega$ such that the divergence theorem can be applied. The integral term, accounting for the viscoelastic damping, describes the dependence of the stress on the strain in the past history. The function g called the memory kernel or the relaxation function. The motivations of problem (1.1) comes from the prototype equation

$$\varrho u_t - \Delta u - \Delta u_{tt} = 0, \tag{1.2}$$

which models several applications in Mechanics. For example, Equation (1.2) describes the extensional vibrations of thin rods [19, Chapter 20] and ion-sound waves [4, Section 6]. However, as observed in [7], in certain cases the material density may depends on small variation of the velocity, that is, $\varrho = \varrho(u_t)$. Then a natural assumption would be

$$\varrho(u_t) = 1 + \epsilon|u_t|^\rho, \quad \rho > 0,$$

where $\epsilon \in \mathbb{R}$ is a small parameter. By simplicity, as explained in [12], we assume $\varrho(u_t) = |u_t|^\rho$. On the other hand, it is important to observe that the classical linearized model yields an integro-differential equation that augments the associated elastic stress tensor with an appropriate memory term which encodes the history of the deformation gradient [10, 24]. From this understanding, the equation (1.1) was obtained by adding memory effects and damping to the model (1.2).

In this paper, we mainly concern the global well-posedness and decay results for the solutions to problem (1.1). We mention there are a lot of works concerning the decay results for evolution equations with memory, see [1, 6, 8, 9, 14, 16, 17, 27, 28] for example. Messaoudi [20] considered problem (1.1) with $\rho = \mu = 0$, and obtained the exponential decay results ([20, Theorem 3.6]) under the assumptions that the memory kernel $g(\cdot)$ satisfies

(G1) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a differential function such that

$$g(0) > 0,\ 1 - \int_0^\infty g(s)ds = l > 0$$

(G2) There exists a differential function ξ such that

$$g'(t) \le -\xi(t)g(t),\ \forall\, t \ge 0;\quad \left|\frac{\xi'(t)}{\xi(t)}\right| \le k,\ \xi(t) > 0,\ \xi'(t) \le 0,\ \forall\, t > 0.$$

Messaoudi and Tatar [23, Theorem 3.1] considered problem (1.1) with $\mu = 1$ and the memory kernel $g(\cdot)$ satisfying (G1) and

$$g'(t) \le -\xi g^p(t), \quad 1 \le p < \frac{3}{2}, \tag{1.3}$$

where $\xi > 0$ is a constant, and got the exponential decay results for $p = 1$ and polynomial decay results for $1 < p < \frac{3}{2}$. Messaoudi and Al-Khulaifi [21] improve [23, Theorem 3.1] by replacing (1.3) with

$$g'(t) \le -\xi(t)g^p(t), \quad 1 \le p < \frac{3}{2}, \tag{1.4}$$

here $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-increasing differential function with $\xi(0) > 0$. Messaoudi and Mustafa [22] got the decay results with the memory kernel $g(\cdot)$ satifying (G1) and (G3), where

(G3) There exists a positive function $H \in C^1(\mathbb{R}^+)$, with $H(0) = 0$, and H is linear or strictly convex C^2 function on $(0, r]$ for some $r < 0$, such that $g'(t) \le -H(g(t))$ for $t \ge 0$.

For more related information on the stability results related to (1.1), we refer the reader to [2, 5, 13, 15, 25, 26] and the references therein.

Recently, Bai et al. [3] considered problem (1.1), the well-posedness ([3, Theorem 2.1]) and the decay results ([3, Theorem 2.4]) were got with the following assumptions:

(A1) ρ is a nonnegative constant that satisfies

$$0 \le \rho \begin{cases} < \infty & \text{if } n = 1, 2; \\ \le \frac{2}{n-2}, & \text{if } n = 3, 4, \cdots. \end{cases}$$

(A2) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a locally absolutely continuous function such that

$$\int_0^\infty g(t)dt < 1,\ g(0) > 0,\ g'(t) \le 0,\ \text{ for a.e. } t \ge 0.$$

(A3) There exists a convex function $H \in C^1(\mathbb{R}^+)$, which is strictly increasing with $H(0) = 0$, such that

$$g'(t) + \xi H(g(t)) \leq 0, \quad \forall\, t \geq 0,$$

where $\xi(\cdot)$ is a nonincreasing positive function with $\lambda := \inf\limits_{t\in[0,\infty)} \xi(t) > 0$.

(A4) Let $y(t)$ be a solution of the following initial value problem of ordinary differential equation

$$y'(t) + \xi(t)H(y(t)) = 0,\; y(0) = g(0),$$

and there exists $\alpha_0 \in (0, 1)$, such that $y^{1-\alpha_0}(\omega(\cdot)) \in L^1(\mathbb{R}^+)$, where $\int_0^{\omega(t)} \xi(\tau)d\tau = t$.

(A5) There exists $\overline{\delta} > 0$ such that $H \in C^2(0, \overline{\delta})$ and

$$\liminf_{x\to 0^+} \left\{x^2H''(x) - xH'(x) + H(x)\right\} \geq 0.$$

In this paper will also investigate problem (1.1). The purposes are two folds:

1. Firstly, for global well-posedness, in [3, Theorem 2.1], the authors showed there exists a unique global solutions $u(t) \in L^\infty(0, \infty; H_0^1(\Omega))$ such that $u_t(t) \in L^\infty(0, \infty; H_0^1(\Omega))$, $u_{tt}(t) \in L^\infty(0, \infty; L^2(\Omega))$. However, the regularity of u makes no sense to the following inner product
$$\left(|u_t|^\rho u_{tt} - \Delta u - \mu\Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds, u\right),$$
especially the term $(\Delta u_{tt}, u)$, which is used throughout the proofs in [3], where $(\cdot, \cdot)$ denotes the standard $L^2(\Omega)$-inner product. To overcome this difficult, we firtly replace u with u_m (it is obvious the above inner product makes sense with u replacing u_m), where u_m it the Galërkin approximation solution; then we study the decay estimates for u_m; finally the decay estimates for u are got by letting $m \to \infty$ with some a-priors estimates. Moreover, our global well-posedness results of this paper ensure $u_{tt}(t) \in L^\infty(0, \infty; H_0^1(\Omega))$ not just $u_{tt}(t) \in L^\infty(0, \infty; L^2(\Omega))$.

2. Secondly, we investigate the stability for problem (1.1) by using more general and weaker assumptions (compared with (A2)-(A5)) on the memory kernel $g(\cdot)$ (see the assumption (H2) below).

Throughout this paper, we make the following assumptions:

(H1) (A1) holds, which implies $H_0^1(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega)$ continuously.

(H2) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-increasing and locally absolutely continuous function such that $g(0) > 0$, $l := 1 - \int_0^\infty g(s)ds > 0$, and $\text{meas}(\Sigma) = 0$, where $\Sigma := \{s \geq 0 : g'(s) = 0\}$.

To simplify our calculation, we introduce the following notations

$$g \circ u := \int_0^t g(t-s)\|u(t) - u(s)\|^2 ds, \quad (u, v) := \int_\Omega uvdx,$$
$$\|u\|_{\rho+2} := \|u\|_{L^{\rho+2}(\Omega)}, \qquad \|u\| := \|u\|_{L^2(\Omega)}.$$

Definition 1.1. (Weak solution) A function $u(x,t)$ is called a global weak solution of (1.1), if

$$u \in L^\infty\left(0, \infty; H_0^1(\Omega)\right), \quad u_t \in L^\infty(0, \infty; H_0^1(\Omega)), \quad u_{tt} \in L^\infty(0, \infty; H_0^1(\Omega)) \tag{1.5}$$

such that for any $\phi(x) \in H_0^1(\Omega)$ and a.e. $t \in (0, \infty)$, there holds

$$(|u_t|^\rho u_{tt}, \phi) + (\nabla u, \nabla \phi) + (\mu \nabla u_{tt}, \nabla \phi) - \left(\int_0^t g(t-s)\nabla u(s)ds, \nabla \phi\right) = 0 \tag{1.6}$$

and

$$u(x, 0) = u_0(x) \text{ in } H_0^1(\Omega),\ u_t(x, 0) = u_1(x) \text{ in } H_0^1(\Omega). \tag{1.7}$$

Remark 1.2. 1. By (1.5), we get $u, u_t \in C([0, \infty); H_0^1(\Omega))$, so (1.7) makes sense.

2. Noting that
$$\frac{\rho}{2(\rho+1)} + \frac{1}{2(\rho+1)} + \frac{1}{2} = 1,$$
by Hölder's inequality and (H1), we get
$$|(|u_t|^\rho u_{tt}, \phi)| \leq \|u_t\|_{2(\rho+1)}^\rho \|u_{tt}\|_{2(\rho+1)} \|\phi\|_2 < \infty,$$
so the first term of (1.6) makes sense, and it is obvious that the other three terms of (1.6) make sense.

Theorem 1.3. *Assume (H1) and (H2) hold. Let $(u_0, u_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$ be given. Then problem* (1.1) *admits a unique global weak solution u. Moreover, for each $t_1 > 0$, there exists a positive constant C depending on $t_1, \rho, \mu, \|\nabla u_0\|, \|\nabla u_1\|$ such that, for any $t \geq t_1$,*

$$\|u_t(t)\|_{\rho+2}^{\rho+2} + \|\nabla u(t)\|^2 + \|\nabla u_t(t)\|^2 \leq \frac{C}{1+t}.$$

The rest of this paper is organized as follows. In Section 2, we study the global well-posednes (the first part of Theorem 1.3); and in Section 3, we study the asymptotical behavior of the solutions (the second part of Theorem 1.3).

2 Global well-posedness

In this section, we study the global well-poseness by using the Faedo-Galërkin method. We first introduce the following Aubin theorem, which can be found in [11, Theorem 1.1.8]:

Lemma 2.1. *Let $T \in (0, \infty), p \in [1, \infty), r \in (1, \infty)$. Assume X, Z and Y are three Banach space such that*

1. *X and Y are reflexive;*
2. *$X \hookrightarrow Z$ compactly and $Z \hookrightarrow Y$ continuous.*

Then $W \hookrightarrow L^p(0, T; Z)$ and $W_1 \hookrightarrow C([0, T]\,; Z)$ compactly, where

$$W := \left\{u \in L^p(0, T; X) : u_t \in L^1(0, T; Y)\right\}, \quad W_1 := \{u \in L^\infty(0, T; X) : u_t \in L^r(0, T; Y)\}\,.$$

Let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \cdots$ be the eigenvalues of $-\Delta$ with homogeneous Dirichlet boundary, and e_i with $\|e_i\| = 1$, $i = 1, 2 \cdots$, be the corresponding eigenfunction of λ_i. It is obvious that $\{e_i\}_{i=1}^{\infty}$ is a completely orthogonal basis of $L^2(\Omega)$ and $H_0^1(\Omega)$, and

$$(e_i, e_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

According to Definition 1.1, the Galërkin approximation solution u_m can be defined as

$$u_m = \sum_{i=1}^{m} \gamma_{im}(t) e_i(x), \quad m = 1, 2, \cdots, \tag{2.1}$$

such that

$$(|u_{mt}|^\rho u_{mtt}, e_i) + (\nabla u_m, \nabla e_i) + (\mu \nabla u_{mtt}, \nabla e_i) - \left(\int_0^t g(t-s) \nabla u_m(s) ds, \nabla e_i \right) = 0 \tag{2.2}$$

for $i = 1, 2, \cdots, m$, and

$$u_m(0) = u_{m0} = \sum_{i=1}^{m} (u_0, e_i)e_i \to u_0 \text{ strongly in } H_0^1(\Omega) \text{ as } m \to \infty, \tag{2.3}$$

$$u_{mt}(0) = u_{m1} = \sum_{i=1}^{m} (u_1, e_i)e_i \to u_1 \text{ strongly in } H_0^1(\Omega) \text{ as } m \to \infty. \tag{2.4}$$

The problem (2.2)-(2.4) is a system of ordinary differential equations with unknown γ_{im}, $i = 1, 2, \cdots, m$. The standard ODE theory shows there exist a solution $\gamma_{im} \in C^2[0, t_m]$ for some $t_m > 0$.

Multiplying the equation (2.2) by $\gamma'_{im}(t)$, and then summing for $i = 1, \cdots, m$, we obtain

$$\underbrace{(|u_{mt}|^\rho u_{mtt}, u_{mt}) + (\nabla u_m, \nabla u_{mt}) + (\mu \nabla u_{mtt}, \nabla u_{mt})}_{=\frac{d}{dt}\left(\frac{1}{\rho+2}\|u_{mt}(t)\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\nabla u_m(t)\|^2 + \frac{\mu}{2}\|\nabla u_{mt}(t)\|^2\right)} = \left(\int_0^t g(t-s)\nabla u_m(s)ds, \nabla u_{mt}\right). \tag{2.5}$$

Noting that

$$\begin{aligned}
&-\left(\int_0^t g(t-s)\nabla u_m(s)ds, \nabla u_{mt}\right) \\
&= \left(\int_0^t g(t-s)(\nabla u_m(t) - \nabla u_m(s))ds, \nabla u_{mt}(t)\right) - \int_0^t g(t-s)\left(\nabla u_m(t), \nabla u_{mt}(t)\right) \\
&= \left(\int_0^t g(t-s)(\nabla u_m(t) - \nabla u_m(s))ds, \nabla u_{mt}(t)\right) - \int_0^t g(s)ds\left(\nabla u_m(t), \nabla u_{mt}(t)\right) \\
&= \frac{1}{2}\frac{d}{dt}\left[(g \circ \nabla u_m)(t)\right] - \frac{1}{2}\left(g' \circ \nabla u_m\right)(t) - \frac{1}{2}\frac{d}{dt}\left[\int_0^t g(s)ds\|\nabla u_m(t)\|^2\right] + \frac{1}{2}g(t)\|\nabla u_m(t)\|^2.
\end{aligned} \tag{2.6}$$

From the (2.5), (2.6) and the assumption (H2), we deduce

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{\rho+2}\|u_{mt}(t)\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u_m(t)\|^2 + \frac{\mu}{2}\|\nabla u_{mt}(t)\|^2 + \frac{1}{2}(g \circ \nabla u_m)(t)\right] \\
&\qquad = \frac{1}{2}(g' \circ \nabla u_m)(t) - g(t)\|\nabla u_m(t)\|^2 \le 0,
\end{aligned}$$

which implies

$$\begin{aligned}
&\frac{1}{\rho+2}\|u_{mt}(t)\|_{\rho+2}^{\rho+2} + \frac{1}{2}\underbrace{\left(1 - \int_0^t g(s)ds\right)}_{\ge l > 0 \text{ by (H2)}}\|\nabla u_m(t)\|^2 + \frac{\mu}{2}\|\nabla u_{mt}(t)\|^2 + \frac{1}{2}(g \circ \nabla u_m)(t) \\
&\qquad \le \frac{1}{\rho+2}\|u_{m1}\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\nabla u_{m0}\|^2 + \frac{\mu}{2}\|\nabla u_{m1}\|^2.
\end{aligned}$$

Then it follows from (2.3) and (2.4) that there exists a positive constant K_1 independent of m such that

$$\|\nabla u_m(t)\|^2 + \|\nabla u_{mt}(t)\|^2 \le K_1, \quad t \in [0, t_m]. \tag{2.7}$$

In view of (H1), (H2) and the above inequality, we get

$$\left\|\int_0^t g(t-s)\nabla u_m(s)ds\right\| \le \sqrt{K_1}\int_0^t g(t-s)ds \le (1-l)\sqrt{K_1}, \quad t\in[0,t_m]. \tag{2.8}$$

and

$$\||u_{mt}|^\rho u_{mt}\|^2_{L^2(\Omega)} = \|u_{mt}(t)\|^{2(\rho+1)}_{L^{2(\rho+1)}} \le c_1^{2(\rho+1)}\|\nabla u_{mt}(t)\|^{2(\rho+1)} \le c_1^{2(\rho+1)}K_1^{\rho+1}, \tag{2.9}$$

where c_1 is the optimal embedding constant of the embedding $H_0^1(\Omega)\hookrightarrow L^{2(\rho+1)}(\Omega)$.

Similarly, multiplying the equation (2.2) by $\gamma''_{im}(t)$, and summing for $i=1,\cdots,m$, we obtain

$$\int_\Omega |u_{mt}(t)|^\rho u^2_{mtt}(t)dx+\mu\|\nabla u_{mtt}(t)\|^2 = -(\nabla u_{mtt}(t),\nabla u_m)+\left(\int_0^t g(t-s)\nabla u_m(s),\nabla u_{mtt}(t)\right).$$

By using the Young's inequality $ab\le\varepsilon a^2+\frac{1}{4\varepsilon}b^2$ with $\varepsilon>0$ arbitrarily and Hölder's inequality, we deduce

$$\int_\Omega |u_{mt}(t)|^\rho u^2_{mtt}(t)dx+\mu\|\nabla u_{mtt}(t)\|^2 \le 2\varepsilon\|\nabla u_{mtt}(t)\|^2+\frac{1}{4\varepsilon}\left[\|\nabla u_m(t)\|^2+\left(\int_0^t g(t-s)\|\nabla u_m(s)\|ds\right)^2\right].$$

Choosing $\varepsilon=\frac{\mu}{4}$ and using the result of (2.7), it follows from the assumption (H2) that

$$\frac{\mu}{2}\|\nabla u_{mtt}(t)\|^2 \le \frac{1}{\mu}K_1+\frac{1}{\mu}(1-l)^2K_1, \quad t\in[0,t_m]. \tag{2.10}$$

By (2.7) and (2.10), we know the local existence time t_m can be extended to ∞, and there holds

$$\|\nabla u_m(t)\|+\left\|\nabla\int_0^t g(t-s)u_m(s)ds\right\|+\|\nabla u_{mt}(t)\|+\|\nabla u_{mtt}(t)\|+\left\||u_{mt}|^{\frac{\varrho}{2}}u_{mtt}\right\|_{L^2(\Omega)} \le K, \tag{2.11}$$

for $t\ge 0$, $m=1,2,\cdots$, and some positive constant K independent of m. Then we get

$$\begin{aligned}
&\{u_m\}_{m=1}^\infty \text{ is bounded in } L^\infty(0,\infty;H_0^1(\Omega)),\\
&\left\{\int_0^t g(t-s)u_m(s)ds\right\}_{m=1}^\infty \text{ is bounded in } L^\infty(0,\infty;H_0^1(\Omega)),\\
&\{u_{mt}\}_{m=1}^\infty \text{ is bounded in } L^\infty(0,\infty;H_0^1(\Omega)),\\
&\{u_{mtt}\}_{m=1}^\infty \text{ is bounded in } L^\infty(0,\infty;H_0^1(\Omega)),\\
&\left\{|u_{mt}(t)|^{\frac{\varrho}{2}}u_{mtt}(t)\right\} \text{ is bounded in } L^\infty\left(0,\infty;L^2(\Omega)\right).
\end{aligned} \tag{2.12}$$

Therefore, there exists a subsequence of $\{u_m\}_{m=1}^\infty$ (still denoted by $\{u_m\}_{m=1}^\infty$), a function $u\in L^\infty(0,\infty;H_0^1(\Omega))$ with $u_t\in L^\infty(0,\infty;H_0^1(\Omega)\cap L^{\rho+2}(\Omega))$ and $u_{tt}\in L^\infty(0,\infty;H_0^1(\Omega))$, and a function

$\chi \in L^{\infty}(0,\infty;L^2)$ such that, as $m \to \infty$,

$$\begin{aligned}
&u_m \rightharpoonup u \text{ weakly star in } L^{\infty}(0,\infty;H_0^1(\Omega)),\\
&\int_0^t g(t-s)u_m(s)ds \rightharpoonup \int_0^t g(t-s)u(s)ds \text{ weakly star in } L^{\infty}(0,\infty;H_0^1(\Omega)),\\
&u_{mt} \rightharpoonup u_t \text{ weakly star in } L^{\infty}(0,\infty;H_0^1(\Omega)),\\
&u_{mtt} \rightharpoonup u_{tt} \text{ weakly star in } L^{\infty}(0,\infty;H_0^1(\Omega)),\\
&\left\{|u_{mt}(t)|^{\frac{\rho}{2}}u_{mtt}(t)\right\} \rightharpoonup \chi \text{ weakly star in } L^{\infty}\left(0,\infty;L^2(\Omega)\right).
\end{aligned} \tag{2.13}$$

Next we show

$$\chi = |u_t|^{\frac{\rho}{2}}u_{tt}. \tag{2.14}$$

In fact, since $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ compactly, it follows from $(2.12)_{2,3}$ and Lemma 2.1 that

$$u_{mt} \to u_t \text{ strongly in } C([0,T];L^2(\Omega)) \tag{2.15}$$

as $m \to \infty$ for any $T \in (0,\infty)$, which implies

$$u_{mt}(x,t) \to u_t(x,t) \text{ for a.e. } (x,t) \in \Omega \times [0,T].$$

Then by [18, Lemma 1.1.3], (2.14) follows.

For any $T \in (0,\infty)$, $\theta(t) \in C_0^{\infty}(0,T)$, we have

$$\begin{aligned}
\int_0^T \int_\Omega |u_{mt}|^{\rho}u_{mtt}\theta(t)e_i dxdt =& \frac{1}{\rho+1}\int_0^T \int_\Omega \frac{d}{dt}\left(|u_{mt}|^{\rho}u_{mt}\right)\theta(t)e_i dxdt\\
=& -\frac{1}{\rho+1}\int_0^T \int_\Omega |u_{mt}|^{\rho}u_{mt}\theta_t e_i dxdt.
\end{aligned}$$

So by multiplying (2.2) with $\theta(t)$ and then integrating from 0 to T, we obtain

$$\begin{aligned}
-\frac{1}{\rho+1}\int_0^T (|u_{mt}|^{\rho}u_{mt},\theta_t(t)e_i)dt + \int_0^T (\nabla u_m,\theta(t)\nabla e_i)dt + \int_0^T (\mu\nabla u_{mtt},\theta(t)\nabla e_i)dt&\\
-\int_0^T \left(\int_0^t g(t-s)\nabla u_m(s)ds,\theta(t)\nabla e_i\right)dt = 0.&
\end{aligned} \tag{2.16}$$

Note that for any $j = 1,2,\cdots,n$

$$\theta_t(t)e_i \in L^1\left(0,T;L^2(\Omega)\right) \text{ and } \theta(t)\frac{\partial e_i}{\partial x_j} \in L^1\left(0,T;H^{-1}(\Omega)\right).$$

In view of (2.13) and (2.14), by letting $m \to \infty$ in (2.16), we get

$$-\frac{1}{\rho+1}\int_0^T (|u_t|^\rho u_t, \theta_t(t)e_i)dt + \int_0^T (\nabla u, \theta(t)\nabla e_i)dt + \int_0^T (\mu\nabla u_{tt}, \theta(t)\nabla e_i)dt \\ - \int_0^T \left(\int_0^t g(t-s)\nabla u(s)ds, \theta(t)\nabla e_i\right) dt = 0.$$

Since $\theta(t) \in C_0^\infty(0,T)$ is arbitrary and $|u_t|^\rho u_t \in L^2(\Omega)$, the above equality implies the distribution derivative of $|u_t|^\rho u_t$ with respect to t belongs to $D'\left(0,T;L^2(\Omega)\right)$ and (note $\{e_i\}_{i=1}^\infty$ is a completely orthogonal basis of $L^2(\Omega)$ and $H_0^1(\Omega)$)

$$\frac{d}{dt}\left(\frac{1}{\rho+1}|u_t|^\rho u_t\right) = |u_t|^\rho u_{tt} = \Delta u + \mu\Delta u_{tt} - \int_0^t g(t-s)\Delta u(s)ds,$$

which implies (1.6) holds.

Next we prove the initial conditions (1.7). By (2.15), we get

$$\lim_{m\to\infty} \|u_{m1} - u_t(0)\|_{L^2(\Omega)} = \lim_{m\to\infty} \|u_{mt}(0) - u_t(0)\|_{L^2(\Omega)} \le \lim_{m\to\infty} \|u_{mt} - u_t\|_{L^\infty(0,T;L^2(\Omega))} = 0,$$

which, together with (2.4) implies $u_t(0) = u_1$. Similar to the proof of (2.15), we have $u_m \to u$ strongly in $C([0,T];L^2(\Omega))$ as $m \to \infty$ for any $T \in (0,\infty)$, so similar argument, we also have $u(0) = u_0$.

Finally, we prove the uniqueness. Let $u = u_1 = u_2$, where u_1, u_2 are any two weak solutions of (1.1). Then we get

$$u \in L^\infty\left(0,\infty;H_0^1(\Omega)\right), \quad u_t \in L^\infty(0,\infty;H_0^1(\Omega)), \quad u_{tt} \in L^\infty(0,\infty;H_0^1(\Omega))$$

such that

$$\begin{cases} (|u_t|^\rho u_{tt}, \phi) + (\nabla u, \nabla\phi) + (\mu\nabla u_{tt}, \nabla\phi) - \left(\int_0^t g(t-s)\nabla u(s)ds, \nabla\phi\right) = 0, \quad \text{a.e. } t \in (0,\infty), \\ u(0) = u_t(0) = 0 \end{cases} \tag{2.17}$$

for any $\phi \in H_0^1(\Omega)$. Let $\phi = u_t$, then by similar argument as (2.7), we get (note by Remark 1.2, $u, u_t \in C([0,\infty);H_0^1(\Omega))$)

$$\|\nabla u(t)\|^2 + \|\nabla u_t(t)\|^2 \equiv 0, \quad t \in [0,\infty),$$

which, together with $(2.17)_2$, implies $u \equiv 0$. So the uniqueness follows.

3 Asymptotical behavior

In this section, we study asymptotical behavior of the solution. Let $u_m(t)$, $t \in [0, \infty)$, be the Galërkin approximation solution defined in (2.1), we define the energy function $E_m(t)$ associated with respect to $u_m(t)$ as

$$E_m(t) = \frac{1}{\rho+2}\|u_{mt}\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u_m\|^2 + \frac{\mu}{2}\|\nabla u_{mt}\|^2 + \frac{1}{2}(g \circ \nabla u_m), \quad t \in [0, \infty). \tag{3.1}$$

Then it is easy to see

$$\frac{dE_m(t)}{dt} = \frac{1}{2}(g' \circ \nabla u_m) - \frac{1}{2}g(t)\|\nabla u_m\|^2 \le 0. \tag{3.2}$$

Let

$$E_0 = \frac{1}{\rho+2}\|u_1\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\nabla u_0\|^2 + \frac{\mu}{2}\|\nabla u_1\|^2 + \frac{1}{2}(g \circ \nabla u_0). \tag{3.3}$$

By (2.3) and (2.4), we have

$$\lim_{m \to \infty} E_m(0) = E_0.$$

So, there exists a positive integer m_0 such that

$$E_m(0) \le E_0 + 1 \tag{3.4}$$

for $m \ge m_0$.

Next we introduce several lemmas, which will be used later. In all the lemmas, we assume (H1) and (H2) hold and $m \ge m_0$. Moreover, for any $\delta \in (0, 1)$, we let

$$G_\delta = \int_0^\infty \frac{g^2(s)}{\delta g(s) - g'(s)}ds \text{ and } J_\delta(t) = \delta g(t) - g'(t). \tag{3.5}$$

Lemma 3.1. *The functional*

$$\chi_{m1}(t) = \frac{1}{\rho+1}\int_\Omega |u_{mt}|^\rho u_{mt} u_m dx + \mu \int_\Omega \nabla u_m \nabla u_{mt} dx,$$

satisfies, for any $0 < \delta < 1$,

$$\chi'_{m1}(t) \le \frac{1}{\rho+1}\|u_{mt}\|_{\rho+2}^{\rho+2} + \mu\|\nabla u_{mt}\|^2 - \frac{l}{2}\|\nabla u_m\|^2 + \frac{1}{2l}G_\delta(J_\delta \circ \nabla u_m)(t),$$

where l *is the constant given in (H2).*

Proof. Multiplying the equation (2.2) by $\gamma_{im}(t)$, and then summing for $i = 1, \cdots, m$, we obtain

$$(|u_{mt}|^\rho u_{mtt}, u_m) + (\nabla u_m, \nabla u_m) + (\mu \nabla u_{mtt}, \nabla u_m) - \left(\int_0^t g(t-s) \nabla u_m(s) ds, \nabla u_m \right) = 0.$$

Then,

$$\begin{aligned}
\chi'_{m1}(t) =&(|u_{mt}|^\rho u_{mtt}, u_m) + \frac{1}{\rho+1}(|u_{mt}|^\rho u_{mt}, u_{mt}) + \mu\|\nabla u_{mt}\|^2 + \mu \int_\Omega \nabla u_m \nabla u_{mtt} dx \\
=& - (\nabla u_m, \nabla u_m) - (\mu \nabla u_{mtt}, \nabla u_m) + \left(\int_0^t g(t-s) \nabla u_m(s) ds, \nabla u_m(t) \right) \\
&+ \frac{1}{\rho+1}\|u_{mt}\|_{\rho+2}^{\rho+2} + \mu\|\nabla u_{mt}\|^2 + \mu \int_\Omega \nabla u_m \nabla u_{mtt} dx \\
=& - \|\nabla u_m(t)\|^2 + \underbrace{\left(\int_0^t g(t-s) \nabla u_m(s) ds, \nabla u_m(t) \right)}_{I_1} + \frac{1}{\rho+1}\|u_{mt}\|_{\rho+2}^{\rho+2} + \mu\|\nabla u_{mt}(t)\|^2.
\end{aligned}$$

By using the Young's inequality, for any $\varepsilon > 0$, we have

$$\begin{aligned}
I_1 =& \left(\int_0^t g(t-s)(\nabla u_m(s) - \nabla u_m(t)) ds, \nabla u_m(t) \right) + \int_0^t g(t-s) ds \, (\nabla u_m(t), \nabla u_m(t)) \\
\le& \frac{\varepsilon}{2}\|\nabla u_m\|^2 + \frac{1}{2\varepsilon} \underbrace{\int_\Omega \left(\int_0^t g(t-s)|\nabla u_m(s) - \nabla u_m(t)| ds \right)^2 dx}_{I_2} + \int_0^t g(s) ds \|\nabla u_m\|^2.
\end{aligned}$$

Using the Cauchy-Schwarz' inequality, we have

$$\begin{aligned}
I_2 =& \int_\Omega \left(\int_0^t \frac{g(t-s)}{\sqrt{\delta g(t-s) - g'(t-s)}} \sqrt{\delta g(t-s) - g'(t-s)} |\nabla u_m(s) - \nabla u_m(t)| ds \right)^2 dx \\
\le& \left(\int_0^t \frac{g^2(s)}{\delta g(s) - g'(s)} ds \right) \int_\Omega \int_0^t [\delta g(t-s) - g'(t-s)] |\nabla u_m(s) - \nabla u_m(t)|^2 ds dx \\
\le& G_\delta (J_\delta \circ \nabla u_m)(t).
\end{aligned}$$

Since $1 - \int_0^t g(s) ds \ge 1 - \int_0^\infty g(s) ds = l$, we choose $\varepsilon = l$, then

$$\begin{aligned}
\chi'_{m1}(t) \le& \frac{1}{\rho+1}\|u_{mt}(t)\|_{\rho+2}^{\rho+2} + \mu\|\nabla u_{mt}(t)\|^2 - l\|\nabla u_m(t)\|^2 + \frac{l}{2}\|\nabla u_m(t)\|^2 + \frac{1}{2l} G_\delta (J_\delta \circ \nabla u_m)(t) \\
=& \frac{1}{\rho+1}\|u_{mt}(t)\|_{\rho+2}^{\rho+2} + \mu\|\nabla u_{mt}(t)\|^2 - \frac{l}{2}\|\nabla u_m(t)\|^2 + \frac{1}{2l} G_\delta (J_\delta \circ \nabla u_m)(t),
\end{aligned}$$

as we desired. □

Lemma 3.2. *For any* $0 < \delta < 1$, $\delta_1, \delta_2, \delta_3 > 0$, *the functional*

$$\chi_{m2}(t) = -\mu \int_0^t g(t-s)(\nabla u_{mt}(t), \nabla u_m(t) - \nabla u_m(s)) ds$$

$$- \frac{1}{\rho+1} \int_\Omega |u_{mt}(t)|^\rho u_{mt}(t) \int_0^t g(t-s)(u_m(t) - u_m(s))dsdx$$

satisfies

$$\begin{aligned}
\chi'_{m2}(t) \leq &- \frac{1}{\rho+1} \int_0^t g(s)ds \|u_{mt}(t)\|_{\rho+2}^{\rho+2} + \frac{\delta_1}{2}\|\nabla u_m(t)\|^2 \\
&+ \left(\mu\delta_3 + c_2\delta_2 - \mu \int_0^t g(s)ds\right) \|\nabla u_{mt}(t)\|^2 \\
&+ \left[\left(\frac{1}{2\delta_1} + 1 + \frac{c}{\delta_3} + \frac{c_3}{\delta_2}\right) G_\delta + \frac{c}{\delta_3} + \frac{c_3}{\delta_2}\right] (J_\delta \circ \nabla u_m)(t),
\end{aligned}$$

where c_1 *is the optimal embedding constant of the embedding* $H_0^1(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega)$, *and* B *is the constant of the Poincaré's inequality*

$$\|\phi\| \leq B\|\nabla\phi\|, \quad \phi \in H_0^1(\Omega),$$

and

$$\begin{aligned}
&c := \max\left\{\frac{\mu}{2}, \frac{\mu\left(g(0)+1-l\right)}{2}\right\}, \\
&c_2 := \frac{c_1^{2(\rho+1)}\left(\frac{2(E_0+1)}{\mu}\right)^{\frac{\rho}{2}}}{\rho+1}, \\
&c_3 := \max\left\{\frac{B^2}{2(\rho+1)}, \frac{B^2(g(0)+1-l)}{2(\rho+1)}\right\}.
\end{aligned} \tag{3.6}$$

Here E_0 *is the constant given in* (3.3).

Proof. Multiplying the equation (2.2) by $\gamma_{im}(t) - \gamma_{im}(s)$, and then summing for $i = 1, \cdots, m$, we obtain

$$\begin{aligned}
(|u_{mt}|^\rho u_{mtt}, u_m(t) - u_m(s)) + (\nabla u_m, \nabla(u_m(t) - u_m(s))) + (\mu\nabla u_{mtt}, \nabla(u_m(t) - u_m(s))) \\
- \left(\int_0^t g(t-s)\nabla u_m(s)ds, \nabla(u_m(t) - u_m(s))\right) = 0.
\end{aligned}$$

Then,

$$\begin{aligned}
\chi'_{m2}(t) = &- \mu \int_0^t g'(t-s)(\nabla u_{mt}(t), \nabla u_m(t) - \nabla u_m(s))ds \\
&- \mu \int_0^t g(t-s)(\nabla u_{mtt}(t), \nabla u_m(t) - \nabla u_m(s))ds \\
&- \mu \int_0^t g(t-s)(\nabla u_{mt}(t), \nabla u_{mt}(t))ds
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\rho+1}\int_\Omega |u_{mt}(t)|^\rho u_{mt}(t)\int_0^t g'(t-s)(u_m(t)-u_m(s))dsdx\\
&-\int_\Omega |u_{mt}(t)|^\rho u_{mtt}(t)\int_0^t g(t-s)(u_m(t)-u_m(s))dsdx\\
&-\frac{1}{\rho+1}\int_\Omega\int_0^t g(t-s)|u_{mt}(t)|^\rho u_{mt}^2(t)dsdx\\
=&\underbrace{\int_0^t g(t-s)(\nabla u_m(t),\nabla u_m(t)-\nabla u_m(s))ds}_{J_1}\\
&\underbrace{-\int_0^t g(t-s)\left(\int_0^t g(t-s)\nabla u_m(s)ds,\nabla u_m(t)-\nabla u_m(s)\right)ds}_{J_2}\\
&\underbrace{-\frac{1}{\rho+1}\int_\Omega |u_{mt}(t)|^\rho u_{mt}(t)\int_0^t g'(t-s)(u_m(t)-u_m(s))dsdx}_{J_3}\\
&-\frac{1}{\rho+1}\int_0^t g(s)ds\|u_{mt}(t)\|_{\rho+2}^{\rho+2}\\
&\underbrace{-\mu\int_0^t g'(t-s)(\nabla u_{mt}(t),\nabla u_m(t)-\nabla u_m(s))ds}_{J_4}-\mu\int_0^t g(s)ds\|\nabla u_{mt}(t)\|^2.
\end{aligned}
$$

Next, we will estimateJ_1, J_2, J_3, J_4. For the J_1, J_2, we have

$$
\begin{aligned}
J_1+J_2=&\int_0^t g(t-s)(\nabla u_m(t),\nabla u_m(t)-\nabla u_m(s))ds\\
&-\int_0^t g(t-s)\left(\int_0^t g(t-s)\nabla u_m(s)ds,\nabla u_m(t)-\nabla u_m(s)\right)ds\\
=&\left(\nabla u_m(t),\int_0^t g(t-s)(\nabla u_m(t)-\nabla u_m(s))ds\right)\\
&-\left(\int_0^t g(t-s)\nabla u_m(s)ds,\int_0^t g(t-s)(\nabla u_m(t)-\nabla u_m(s))ds\right)\\
&-\int_0^t g(s)ds\left(\nabla u_m(t),\int_0^t g(t-s)(\nabla u_m(t)-\nabla u_m(s))ds\right)\\
&+\int_0^t g(s)ds\left(\nabla u_m(t),\int_0^t g(t-s)(\nabla u_m(t)-\nabla u_m(s))ds\right)\\
=&\left(1-\int_0^t g(s)ds\right)\left(\nabla u_m(t),\int_0^t g(t-s)(\nabla u_m(t)-\nabla u_m(s))ds\right)\\
&+\left\|\int_0^t g(t-s)(\nabla u_m(t)-\nabla u_m(s))ds\right\|^2.
\end{aligned}
$$

Since $g(t) > 0$, $\int_0^\infty g(s)ds < 1$ (see the assumption (H2)), for any $\delta_1 > 0$, using Young's inequality and Cauchy-Schwarz' inequality, similar to the estimate of I_2 in the proof of Lemma 3.1, we have

$$\begin{aligned}
&\left(1-\int_0^t g(s)ds\right)\left(\nabla u_m(t), \int_0^t g(t-s)(\nabla u_m(t)-\nabla u_m(s))ds\right)\\
\le&\frac{\delta_1}{2}\|\nabla u_m(t)\|^2+\frac{1}{2\delta_1}\int_\Omega\left(\int_0^t g(t-s)|\nabla u_m(s)-\nabla u_m(t)|ds\right)^2dx\\
\le&\frac{\delta_1}{2}\|\nabla u_m(t)\|^2+\frac{1}{2\delta_1}G_\delta(J_\delta\circ\nabla u_m)(t)
\end{aligned}$$

and

$$\begin{aligned}
\left\|\int_0^t g(t-s)(\nabla u_m(t)-\nabla u_m(s))ds\right\|^2=&\int_\Omega\left(\int_0^t g(t-s)|\nabla u_m(s)-\nabla u_m(t)|ds\right)^2dx\\
\le&G_\delta(J_\delta\circ\nabla u_m)(t).
\end{aligned}$$

The above estimates imply

$$|J_1+J_2|\le\frac{\delta_1}{2}\|\nabla u_m(t)\|^2+\left(\frac{1}{2\delta_1}+1\right)G_\delta(J_\delta\circ\nabla u_m)(t). \tag{3.7}$$

Since $g'(t-s)=\delta g(t-s)-J_\delta(t-s)$, we have

$$\begin{aligned}
|J_3|\le&\frac{1}{\rho+1}\left|\left(|u_{mt}|^{\rho+1}, \int_0^t \delta g(t-s)|u_m(t)-u_m(s)|ds\right)\right|\\
&+\frac{1}{\rho+1}\left|\left(|u_{mt}|^{\rho+1}, \int_0^t J_\delta(t-s)|u_m(t)-u_m(s)|ds\right)\right|.
\end{aligned} \tag{3.8}$$

Since c_1 is the optimal embedding constant of the embedding $H_0^1(\Omega)\hookrightarrow L^{2(\rho+1)}(\Omega)$, by (3.2) and (3.4), we deduce

$$\begin{aligned}
\|u_{mt}\|_{2(\rho+1)}^{2(\rho+1)}\le&\, c_1^{2(\rho+1)}\|\nabla u_{mt}\|_{L^2}^{2(\rho+1)}\\
=&\, c_1^{2(\rho+1)}\left(\int_\Omega(\nabla u_{mt})^2dx\right)\left(\int_\Omega(\nabla u_{mt})^2dx\right)^\rho\\
\le&\, c_1^{2(\rho+1)}\left(\frac{2(E_0+1)}{\mu}\right)^{\frac{\rho}{2}}\|\nabla u_{mt}(t)\|^2.
\end{aligned} \tag{3.9}$$

Then by using the Young's inequality, it follows from (3.8) that, for any $\delta_2 > 0$,

$$\begin{aligned}
|J_3|\le&\frac{1}{\rho+1}\left[\frac{\delta_2}{2}\|u_{mt}(t)\|_{2(\rho+1)}^{2(\rho+1)}+\frac{\delta^2}{2\delta_2}\int_\Omega\left(\int_0^t g(t-s)|u_m(t)-u_m(s)|ds\right)^2dx\right]\\
&+\frac{1}{\rho+1}\left[\frac{\delta_2}{2}\|u_{mt}(t)\|_{2(\rho+1)}^{2(\rho+1)}+\frac{1}{2\delta_2}\int_\Omega\left(\int_0^t J_\delta(t-s)|u_m(t)-u_m(s)|ds\right)^2dx\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\delta_2 c_1^{2(\rho+1)} \left(\frac{2(E_0+1)}{\mu}\right)^{\frac{\rho}{2}}}{\rho+1} \|\nabla u_{mt}(t)\|^2 + \frac{\delta^2}{2(\rho+1)\delta_2} G_\delta(J_\delta \circ u_m)(t) \\
&\quad + \frac{1}{2(\rho+1)\delta_2} \int_\Omega \left(\int_0^t \sqrt{J_\delta(t-s)}\sqrt{J_\delta(t-s)} |u_m(t) - u_m(s)| ds \right)^2 dx \\
&\leq \frac{\delta_2 c_1^{2(\rho+1)} \left(\frac{2(E_0+1)}{\mu}\right)^{\frac{\rho}{2}}}{\rho+1} \|\nabla u_{mt}(t)\|^2 + \frac{\delta^2}{2(\rho+1)\delta_2} G_\delta(J_\delta \circ u_m)(t) + \frac{\int_0^t J_\delta(t-s)ds}{2(\rho+1)\delta_2} (J_\delta \circ u_m)(t) \\
&\leq \frac{\delta_2 c_1^{2(\rho+1)} \left(\frac{2(E_0+1)}{\mu}\right)^{\frac{\rho}{2}}}{\rho+1} \|\nabla u_{mt}(t)\|^2 + \frac{B^2\delta^2}{2(\rho+1)\delta_2} G_\delta(J_\delta \circ \nabla u_m)(t) + \frac{B^2 \int_0^t J_\delta(t-s)ds}{2(\rho+1)\delta_2} (J_\delta \circ \nabla u_m)(t).
\end{aligned}
$$

Noticing (H2) and $0 < \delta < 1$, we have

$$
\begin{aligned}
\int_0^t J_\delta(t-s)ds = \int_0^t J_\delta(s)ds &= \int_0^t (\delta g(s) - g'(s))ds \\
&\leq \int_0^\infty g(s)ds + g(0) - g(t) \leq g(0) + 1 - l.
\end{aligned}
$$

Then it follows

$$
|J_3| \leq c_2\delta_2 \|\nabla u_{mt}(t)\|^2 + \left(\frac{c_3\delta^2}{\delta_2} G_\delta + \frac{c_3}{\delta_2} \right) (J_\delta \circ \nabla u_m)(t), \tag{3.10}
$$

where c_2 and c_3 are the two constants defined in (3.6).

Since $g'(t-s) = \delta g(t-s) - J_\delta(t-s)$, we have

$$
\begin{aligned}
J_4 =& -\mu \int_0^t \delta g(t-s)(\nabla u_{mt}(t), \nabla u_m(t) - \nabla u_m(s))ds \\
&+ \mu \int_0^t J_\delta(t-s)(\nabla u_{mt}(t), \nabla u_m(t) - \nabla u_m(s))ds \\
=& -\mu \left(\nabla u_{mt}(t), \int_0^t \delta g(t-s)(\nabla u_m(t) - \nabla u_m(s))ds \right) \\
&+ \mu \left(\nabla u_{mt}(t), \int_0^t J_\delta(t-s)(\nabla u_m(t) - \nabla u_m(s))ds \right),
\end{aligned}
$$

Applying Young's inequality, for any $\delta_3 > 0$, we obtain

$$
\begin{aligned}
|J_4| \leq& \left[\frac{\mu\delta_3}{2} \|\nabla u_{mt}(t)\|^2 + \frac{\mu\delta^2}{2\delta_3} \int_\Omega \left(\int_0^t g(t-s) |\nabla u_m(t) - \nabla u_m(s)| ds \right)^2 dx \right] \\
&+ \left[\frac{\mu\delta_3}{2} \|\nabla u_{mt}(t)\|^2 + \frac{\mu}{2\delta_3} \int_\Omega \left(\int_0^t \sqrt{J_\delta(t-s)}\sqrt{J_\delta(t-s)} |\nabla u_m(t) - \nabla u_m(s)| ds \right)^2 dx \right] \\
\leq& \mu\delta_3 \|\nabla u_{mt}(t)\|^2 + \frac{\mu\delta^2}{2\delta_3} G_\delta(J_\delta \circ \nabla u_m)(t) + \frac{\mu}{2\delta_3} \int_0^t J_\delta(t-s)ds (J_\delta \circ \nabla u_m)(t),
\end{aligned}
$$

which implies,

$$|J_4| \le \mu\delta_3\|\nabla u_{mt}(t)\|^2 + \left(\frac{c\delta^2}{\delta_3}G_\delta + \frac{c}{\delta_3}\right)(J_\delta \circ \nabla u_m)(t), \tag{3.11}$$

Where c is the constant defined in (3.6).

Then it follows from (3.7), (3.10) and (3.11) that

$$\begin{aligned}\chi'_{m2}(t) \le & -\frac{1}{\rho+1}\int_0^t g(s)ds\|u_{mt}(t)\|_{\rho+2}^{\rho+2} + \frac{\delta_1}{2}\|\nabla u_m(t)\|^2 + \left(\mu\delta_3 + c_2\delta_2 - \mu\int_0^t g(s)ds\right)\|\nabla u_{mt}(t)\|^2 \\ & + \left[\left(\frac{1}{2\delta_1}+1\right)G_\delta + \left(\frac{c\delta^2}{\delta_3}G_\delta + \frac{c}{\delta_3}\right) + \left(\frac{c_3\delta^2}{\delta_2}G_\delta + \frac{c_3}{\delta_2}\right)\right](J_\delta \circ \nabla u_m)(t).\end{aligned}$$

As $0 < \delta < 1$, we have

$$\begin{aligned}\chi'_{m2}(t) \le & -\frac{1}{\rho+1}\int_0^t g(s)ds\|u_{mt}(t)\|_{\rho+2}^{\rho+2} + \frac{\delta_1}{2}\|\nabla u_m(t)\|^2 + \left(\mu\delta_3 + c_2\delta_2 - \mu\int_0^t g(s)ds\right)\|\nabla u_{mt}(t)\|^2 \\ & + \left[\left(\frac{1}{2\delta_1}+1+\frac{c}{\delta_3}+\frac{c_3}{\delta_2}\right)G_\delta + \frac{c}{\delta_3} + \frac{c_3}{\delta_2}\right](J_\delta \circ \nabla u_m)(t),\end{aligned}$$

as we desired. □

For any constants $M_1, M_2 > 0$, we let

$$\Phi_m(t) := ME_m(t) + M_1\chi_{m1}(t) + M_2\chi_{m2}(t). \tag{3.12}$$

Then we have the following lemma:

Lemma 3.3. *Let $M \ge \kappa + 1$, where*

$$\kappa := \left(1 + \frac{c_2}{\mu} + \frac{\mu}{l} + \frac{B^2}{l(\rho+1)}\right)M_1 + \left(1 + \frac{c_2}{\mu} + \mu(1-l) + \frac{(1-l)B^2}{\rho+1}\right)M_2, \tag{3.13}$$

then $\Phi_m(t) \sim E_m(t)$, i.e.,

$$E_m(t) \le \Phi_m(t) \le (M+\kappa)E_m(t).$$

Here $l \in (0,1)$ is the constant given in (H2), c_2 and B are the two positive constants given in Lemma 3.2.

Proof. Applying Young's inequality, Lemmas 3.1 and 3.2, and the inequality (3.9), we get (note the definitions of the constants c_2 and B in Lemma 3.2)

$$\begin{aligned}|\Phi_m(t) - ME_m(t)| \le & M_1|\chi_{m1}(t)| + M_2|\chi_{m2}(t)| \\ \le & \frac{M_1}{2(\rho+1)}\|u_{mt}\|_{2(\rho+1)}^{2(\rho+1)} + \frac{M_1}{2(\rho+1)}\|u_m\|^2 + \frac{\mu M_1}{2}\|\nabla u_m\|^2 + \frac{\mu M_1}{2}\|\nabla u_{mt}\|^2\end{aligned}$$

$$
\begin{aligned}
&+\frac{\mu M_2}{2}\|\nabla u_{mt}\|^2+\frac{\mu M_2}{2}\int_\Omega\left(\int_0^t g(t-s)(\nabla u_m(t)-\nabla u_m(s))ds\right)^2dx\\
&+\frac{M_2}{2(\rho+1)}\|u_{mt}\|_{2(\rho+1)}^{2(\rho+1)}+\frac{M_2}{2(\rho+1)}\int_\Omega\left(\int_0^t g(t-s)(u_m(t)-u_m(s))ds\right)^2dx\\
\leq&\frac{M_1c_2}{2}\|\nabla u_{mt}\|^2+\frac{M_1B^2}{2(\rho+1)}\|\nabla u_m\|^2+\frac{\mu M_1}{2}\|\nabla u_m\|^2+\frac{\mu M_1}{2}\|\nabla u_{mt}\|^2\\
&+\frac{\mu M_2}{2}\|\nabla u_{mt}\|^2+\frac{\mu M_2}{2}\int_0^t g(s)ds\,(g\circ\nabla u_m)(t)+\frac{M_2c_2}{2}\|\nabla u_{mt}\|^2\\
&+\frac{M_2}{2(\rho+1)}\int_0^t g(s)ds\,(g\circ u_m)(t)\\
\leq&\frac{M_1c_2}{2}\|\nabla u_{mt}\|^2+\frac{M_1B^2}{2(\rho+1)}\|\nabla u_m\|^2+\frac{\mu M_1}{2}\|\nabla u_m\|^2+\frac{\mu M_1}{2}\|\nabla u_{mt}\|^2\\
&+\frac{\mu M_2}{2}\|\nabla u_{mt}\|^2+\frac{\mu M_2(1-l)}{2}(g\circ\nabla u_m)(t)+\frac{M_2c_2}{2}\|\nabla u_{mt}\|^2\\
&+\frac{M_2B^2(1-l)}{2(\rho+1)}(g\circ\nabla u_m)(t)\\
\leq&\left(\frac{(M_1+M_2)c_2}{2}+\frac{\mu(M_1+M_2)}{2}\right)\|\nabla u_{mt}\|^2\\
&+\left(\frac{\mu M_1}{2}+\frac{M_1B^2}{2(\rho+1)}\right)\|\nabla u_m\|^2+\left(\frac{\mu M_2(1-l)}{2}+\frac{M_2B^2(1-l)}{2(\rho+1)}\right)(g\circ\nabla u_m)(t)\\
\leq&\left(\frac{(M_1+M_2)c_2}{\mu}+M_1+M_2\right)E_m(t)\\
&+\left(\frac{\mu M_1}{l}+\frac{M_1B^2}{l(\rho+1)}\right)E_m(t)+\left(\mu M_2(1-l)+\frac{M_2B^2(1-l)}{\rho+1}\right)E_m(t)\\
=&\kappa E_m(t)
\end{aligned}
$$

with κ been defined in (3.13). Then the desired results follows. □

Lemma 3.4. *The functional*

$$\chi_{m3}(t)=\int_\Omega\int_0^t h(t-s)|\nabla u_m(s)|^2dsdx,$$

satisfies

$$\chi_{m3}'(t)\leq 3(1-l)\|\nabla u_m\|^2-\frac{1}{2}(g\circ\nabla u_m)(t),$$

and, for any $t_1\geq 0$,

$$\chi_{m3}(t_1)\leq\frac{2(1-l)}{l}(E_0+1)t_1,$$

where $h(t)=\int_t^\infty g(s)ds$ and E_0 is the constant given in (3.3).

Proof. Applying Young's inequality and $h'(t) = -g(t)$, we get

$$\begin{aligned}
\chi'_{m3}(t) =& h(0)\int_\Omega |\nabla u_m(t)|^2 dx - \int_\Omega\int_0^t g(t-s)|\nabla u_m(s)|^2 dsdx \\
=& h(0)\int_\Omega |\nabla u_m(t)|^2 dx - \int_\Omega\int_0^t g(t-s)|(\nabla u_m(s)-\nabla u_m(t)) + \nabla u_m(t)|^2 dsdx \\
=& \underbrace{-\int_\Omega\int_0^t g(t-s)|\nabla u_m(s)-\nabla u_m(t)|^2 dsdx}_{=-(g\circ\nabla u_m)(t)} \\
& - 2\int_\Omega \nabla u_m(t)\int_0^t g(t-s)(\nabla u_m(s)-\nabla u_m(t))dsdx \\
& - \int_\Omega\int_0^t g(t-s)|\nabla u_m(t)|^2 dsdx + h(0)\int_\Omega |\nabla u_m(t)|^2 dx.
\end{aligned}$$

Noticing that

$$\begin{aligned}
& -\int_\Omega\int_0^t g(t-s)|\nabla u_m(t)|^2 dsdx + h(0)\int_\Omega |\nabla u_m(t)|^2 dx \\
=& -\int_0^t g(s)ds\int_\Omega |\nabla u_m(t)|^2 dx + \int_0^\infty g(s)ds\int_\Omega |\nabla u_m(t)|^2 dx = h(t)\int_\Omega |\nabla u_m(t)|^2 dx,
\end{aligned}$$

and for any $\varepsilon > 0$,

$$\begin{aligned}
& - 2\int_\Omega \nabla u_m(t)\int_0^t g(t-s)(\nabla u_m(s)-\nabla u_m(t))dsdx \\
\leq & 2\varepsilon\|\nabla u_m(t)\|^2 + \frac{1}{2\varepsilon}\int_0^t g(s)ds\int_\Omega\int_0^t g(t-s)|\nabla u_m(s)-\nabla u_m(t)|^2 dsdx,
\end{aligned}$$

it follows (note the assumption (H2) and $h(t) \leq h(0) = \int_0^\infty g(s)ds = 1-l$)

$$\begin{aligned}
\chi'_{m3}(t) =& \left(\frac{1}{2\varepsilon}\int_0^t g(s)ds - 1\right)(g\circ\nabla u_m)(t) + (2\varepsilon + h(t))\|\nabla u_m(t)\|^2 \\
\leq & \left(\frac{1}{2\varepsilon}(1-l) - 1\right)(g\circ\nabla u_m)(t) + (2\varepsilon + 1 - l)\|\nabla u_m(t)\|^2.
\end{aligned}$$

By choosing $\varepsilon = 1-l$ in the above inequality, we obtain

$$\chi'_{m3}(t) \leq 3(1-l)\|\nabla u_m(t)\|^2 - \frac{1}{2}(g\circ\nabla u_m)(t).$$

Meanwhile, using (H2), (3.1), (3.2), and (3.4), we obtain

$$\begin{aligned}
\chi_{m3}(t_1) =& \int_\Omega\int_0^{t_1} h(t_1-s)|\nabla u_m(s)|^2 dsdx \\
\leq & \left(\int_0^\infty g(s)ds\right)\int_\Omega\int_0^{t_1} |\nabla u_m(s)|^2 dsdx \\
\leq & (1-l)\int_0^{t_1}\|\nabla u_m(s)\|^2 ds \leq \frac{2(1-l)}{l}(E_0+1)t_1.
\end{aligned}$$

As we desired. ☐

Now we are ready to study the asymptotical behavior of the solution. Let $\Phi_m(t)$ be the functional defined in (3.12), where M_1, M_2 are two positive constant be determined later, and $M \geq \kappa+1$, where $\kappa = \kappa(M_1, M_2)$ is the positive constant given in (3.13).

For any $t_1 > 0$, we let

$$g_1 := \int_0^{t_1} g(s)ds,$$

which is positive. Then it follows from (3.2), Lemma 3.1, and Lemma 3.2 that (note that by (3.5), $g'(t) = \delta g(t) - J_\delta(t)$), for $t \geq t_1$,

$$\begin{aligned}
\Phi_m'(t) \leq & M\left(\frac{1}{2}(g' \circ \nabla u_m) - \frac{1}{2}g(t)\|\nabla u_m\|^2\right) \\
& + M_1\left(\frac{1}{\rho+1}\|u_{mt}\|_{\rho+2}^{\rho+2} + \mu\|\nabla u_{mt}\|^2 - \frac{l}{2}\|\nabla u_m\|^2 + \frac{1}{2l}G_\delta(J_\delta \circ \nabla u_m)(t)\right) \\
& + M_2\left(-\frac{1}{\rho+1}\int_0^t g(s)ds\|u_{mt}(t)\|_{\rho+2}^{\rho+2} + \frac{\delta_1}{2}\|\nabla u_m(t)\|^2\right) \\
& + M_2\left[\left(\mu\delta_3 + c_2\delta_2 - \mu\int_0^t g(s)ds\right)\|\nabla u_{mt}(t)\|^2\right] \\
& + M_2\left[\left(\left(\frac{1}{2\delta_1} + 1 + \frac{c}{\delta_3} + \frac{c_3}{\delta_2}\right)G_\delta + \frac{c}{\delta_3} + \frac{c_3}{\delta_2}\right)(J_\delta \circ \nabla u_m)(t)\right] \\
\leq & \frac{\delta M}{2}(g \circ \nabla u_m)(t) - \frac{M}{2}(J_\delta \circ \nabla u_m)(t) \\
& + \frac{M_1}{\rho+1}\|u_{mt}(t)\|_{\rho+2}^{\rho+2} + \mu M_1\|\nabla u_{mt}(t)\|^2 - \frac{lM_1}{2}\|\nabla u_m(t)\|^2 + \frac{M_1}{2l}G_\delta(J_\delta \circ \nabla u_m)(t) \\
& - \frac{g_1 M_2}{\rho+1}\|u_{mt}(t)\|_{\rho+2}^{\rho+2} + \frac{\delta_1 M_2}{2}\|\nabla u_m(t)\|^2 + (\mu\delta_3 M_2 + c_2\delta_2 M_2 - \mu g_1 M_2)\|\nabla u_{mt}(t)\|^2 \\
& + \left[\left(\frac{c}{\delta_3} + \frac{c_3}{\delta_2}\right)M_2 + \left(\frac{1}{2\delta_1} + 1 + \frac{c}{\delta_3} + \frac{c_3}{\delta_2}\right)M_2 G_\delta\right](J_\delta \circ \nabla u_m)(t) \\
\leq & -\left(\frac{M_2 g_1 - M_1}{\rho+1}\right)\|u_{mt}(t)\|_{\rho+2}^{\rho+2} - \left(\frac{lM_1 - \delta_1 M_2}{2}\right)\|\nabla u_m(t)\|^2 \\
& - (\mu g_1 M_2 - \mu M_1 - \mu\delta_3 M_2 - c_2\delta_2 M_2)\|\nabla u_{mt}(t)\|^2 \\
& - \left[\frac{M}{2} - \left(\frac{c}{\delta_3} + \frac{c_3}{\delta_2}\right)M_2 - \left(\frac{M_2}{2\delta_1} + M_2 + \frac{cM_2}{\delta_3} + \frac{c_3 M_2}{\delta_2} + \frac{M_1}{2l}\right)G_\delta\right](J_\delta \circ \nabla u_m)(t) \\
& + \frac{\delta M}{2}(g \circ \nabla u_m)(t).
\end{aligned}$$

Now, taking $\delta_1 = \delta_2 = \delta_3 = \frac{l}{2M_2}$, we get

$$\Phi_m'(t) \leq -\left(\frac{M_2 g_1 - M_1}{\rho+1}\right)\|u_{mt}(t)\|_{\rho+2}^{\rho+2} - \left(\frac{lM_1 - \frac{l}{2}}{2}\right)\|\nabla u_m(t)\|^2$$

$$
\begin{aligned}
&-\left(\mu g_1 M_2 - \mu M_1 - \frac{\mu l}{2} - \frac{c_2 l}{2}\right)\|\nabla u_{mt}(t)\|^2 \\
&-\left[\frac{M}{2} - \left(\frac{2c}{l} + \frac{2c_3}{l}\right)M_2^2 - \left(\frac{M_2^2}{l} + M_2 + \frac{2cM_2^2}{l} + \frac{2c_3M_2^2}{l} + \frac{M_1}{2l}\right)G_\delta\right](J_\delta \circ \nabla u_m)(t) \\
&+\frac{\delta M}{2}(g \circ \nabla u_m)(t).
\end{aligned} \tag{3.14}
$$

Let

$$
\begin{aligned}
M_1 &:= \frac{16(1-l) + \frac{l}{2}}{l} + 1, \\
M_2 &:= \frac{1 + M_1 + \frac{l}{2} + \frac{c_2 l}{2\mu}}{g_1} + 1.
\end{aligned}
$$

We have

$$
\begin{aligned}
&lM_1 - \frac{l}{2} > 16(1-l), \\
&M_2 g_1 - M_1 > 1, \\
&\mu g_1 M_2 - \mu M_1 - \frac{\mu l}{2} - \frac{c_2 l}{2} > \mu.
\end{aligned} \tag{3.15}
$$

Since $\lim\limits_{\delta\to 0} \frac{\delta g^2(s)}{\delta g(s) - g'(s)} = 0$ for a.e. $s \in (0,\infty)$ and $\frac{\delta g^2(s)}{\delta g(s) - g'(s)} \leq g(s) \in L^1(0,\infty)$, we can apply the Lebesegue's Dominated Convergence Theorem to get

$$
\lim_{\delta\to 0} \delta G_\delta = \lim_{\delta\to 0} \int_0^\infty \frac{\delta g^2(s)}{\delta g(s) - g'(s)} ds = 0.
$$

Therefore, there exist $0 < \delta_0 < 1$ so that if $0 < \delta \leq \delta_0$,

$$
\delta G_\delta < \frac{1}{16\left(\frac{M_2^2}{l} + M_2 + \frac{2cM_2^2}{l} + \frac{2c_3M_2^2}{l} + \frac{M_1}{2l}\right)}.
$$

By choosing M large enough such that $M \geq \kappa + 1$ (so that Lemma 3.3 is satisfied), and

$$
\frac{M}{4} - \left(\frac{2c}{l} + \frac{2c_3}{l}\right)M_2^2 > 0 \text{ and } \frac{1}{4M} \leq \delta_0.
$$

Hence, by taking $\delta = \frac{1}{4M}$, we get

$$
\begin{aligned}
&\frac{M}{2} - \left(\frac{2c}{l} + \frac{2c_3}{l}\right)M_2^2 - \left(\frac{M_2^2}{l} + M_2 + \frac{2cM_2^2}{l} + \frac{2c_3M_2^2}{l} + \frac{M_1}{2l}\right)G_\delta \\
&= \left[\frac{M}{4} - \left(\frac{2c}{l} + \frac{2c_3}{l}\right)M_2^2\right] + \frac{M}{4} - \left(\frac{M_2^2}{l} + M_2 + \frac{2cM_2^2}{l} + \frac{2c_3M_2^2}{l} + \frac{M_1}{2l}\right)G_\delta \\
&\geq \left[\frac{M}{4} - \left(\frac{2c}{l} + \frac{2c_3}{l}\right)M_2^2\right] + \underbrace{\frac{M}{4} - \frac{1}{16\delta G_\delta}G_\delta}_{=0} \\
&>0.
\end{aligned} \tag{3.16}
$$

So, we get from (3.14)-(3.16) that

$$\Phi'_m(t) \le -\frac{1}{\rho+1}\|u_{mt}(t)\|_{\rho+2}^{\rho+2} - \mu\|\nabla u_{mt}(t)\|^2 - 8(1-l)\|\nabla u_m(t)\|^2 + \frac{1}{8}(g\circ\nabla u_m)(t) \tag{3.17}$$

for $t \ge t_1$.

Next, we let

$$L_m(t) := \Phi_m(t) + \chi_{m3}(t), \quad t \ge t_1,$$

with $\Phi_m(t)$ and $\chi_{m3}(t)$ been defined in (3.12) and Lemma 3.4, respectively, where M, M_1, and M_2 are the three positive constants choosing above. Then, by (3.17) and Lemma 3.4, we have

$$\begin{aligned}
L'_m(t) =& \Phi'_m(t) + \chi'_{m3}(t)\\
\le& -\frac{1}{\rho+1}\|u_{mt}(t)\|_{\rho+2}^{\rho+2} - \mu\|\nabla u_{mt}(t)\|^2 - 8(1-l)\|\nabla u_m(t)\|^2 + \frac{1}{8}(g\circ\nabla u_m)(t)\\
&+ 3(1-l)\|\nabla u_m(t)\|^2 - \frac{1}{2}(g\circ\nabla u_m)(t)\\
=& -\frac{1}{\rho+1}\|u_{mt}(t)\|_{\rho+2}^{\rho+2} - \mu\|\nabla u_{mt}(t)\|^2 - 5(1-l)\|\nabla u_m(t)\|^2 - \frac{3}{8}(g\circ\nabla u_m)(t)\\
=& -\frac{\rho+2}{\rho+1}\frac{1}{\rho+2}\|u_{mt}(t)\|_{\rho+2}^{\rho+2} - 10(1-l)\frac{1}{2}\|\nabla u_m(t)\|^2 - 2\frac{\mu}{2}\|\nabla u_{mt}(t)\|^2 - \frac{3}{4}\times\frac{1}{2}(g\circ\nabla u_m)(t).
\end{aligned}$$

Note (see (3.1))

$$E_m(t) \le \frac{1}{\rho+2}\|u_{mt}\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\nabla u_m\|^2 + \frac{\mu}{2}\|\nabla u_{mt}\|^2 + \frac{1}{2}(g\circ\nabla u_m),$$

by taking

$$k_1 = \min\left\{\frac{\rho+2}{\rho+1},\ 10(1-l),\ \frac{3}{4}\right\},$$

we get

$$L'_m(t) \le -k_1 E_m(t), \quad t \ge t_1.$$

Note $\chi_{m3}(t) \ge 0$ (see Lemma 3.4), we get from Lemma 3.3 and (3.1) that $L_m(t) \ge 0$. Then integrating the above inequality from t_1 to t, we get from Lemma 3.3, (3.2), (3.4), and Lemma 3.4 that

$$\begin{aligned}
k_1\int_{t_1}^t E_m(s)ds \le& L_m(t_1)) - L_m(t)\\
\le& L_m(t_1))\\
=& \Phi_m(t_1) + \chi_{m3}(t_1)
\end{aligned}$$

$$\leq (M+\kappa)E_m(t_1) + \chi_{m3}(t_1)$$
$$\leq (M+\kappa)E_m(0) + +\chi_{m3}(t_1)$$
$$\leq (M+\kappa)(E_0+1) + \frac{2(1-l)}{l}(E_0+1)t_1 =: \varrho,$$

which means

$$\int_{t_1}^{\infty} E_m(s)ds \leq \frac{\varrho}{k_1}. \tag{3.18}$$

Since for all $t \geq t_1$, we have (note (3.2))

$$\frac{d[(t+1)E_m(t)]}{dt} = (t+1)E_m'(t) + E_m(t) \leq E_m(t),$$

it follows from $E_m(t) \geq 0$ and (3.18) that

$$(t+1)E_m(t) - (t_1+1)E_m(t_1) \leq \int_{t_1}^{\infty} E_m(s)ds \leq \frac{\varrho}{k_1}.$$

Then we get from (3.2) and (3.4) that

$$E_m(t) \leq \frac{C}{1+t} \tag{3.19}$$

for $t \geq t_1$ with

$$C = \frac{\varrho}{k_1} + (t_1+1)(E_0+1).$$

Then it follows from (H2), (3.1) and (3.19) that

$$\frac{1}{\rho+2}\|u_{mt}\|_{\rho+2}^{\rho+2} + \frac{l}{2}\|\nabla u_m\|^2 + \frac{\mu}{2}\|\nabla u_{mt}\|^2 \leq \frac{C}{1+t}, \quad t \geq t_1. \tag{3.20}$$

By (H1), we get $H_0^1(\Omega) \hookrightarrow L^{\rho+2}(\Omega)$ compactly and $L^{\rho+2}(\Omega) \hookrightarrow L^2(\Omega)$ continuous. Then it follows from (2.12), (2.13) and Lemma 2.1 that for any $T \in (t_1, \infty)$,

$$u_{mt} \to u_t \text{ strongly in } C\left([t_1,T]; L^{\rho+2}(\Omega)\right)$$

as $m \to \infty$. Then by the arbitrariness of T, we obtain for any $t \geq t_1$,

$$\lim_{m\to\infty} \|u_{mt}(t)\|_{\rho+2} = \|u_t(t)\|_{\rho+2},$$

which, together with (3.20), implies

$$\|u_t\|_{\rho+2}^{\rho+2} \leq \frac{C(\rho+2)}{t+1}, \quad t \geq t_1. \tag{3.21}$$

For any $T \in (t_1, T)$ and $p \in (1, \infty)$, by (2.12), (2.13), we get, as $m \to \infty$,

$$\nabla u_m(t) \rightharpoonup \nabla u \text{ and } \nabla u_{mt}(t) \rightharpoonup \nabla u_t \text{ weakly in } L^p\left(t_1, T; L^2(\Omega)\right),$$

which implies

$$\begin{aligned} \|\nabla u\|_{L^p(t_1,T;L^2(\Omega))} &\leq \liminf_{m\to\infty} \|\nabla u_m(t)\|_{L^p(t_1,T;L^2(\Omega))}, \\ \|\nabla u_t\|_{L^p(t_1,T;L^2(\Omega))} &\leq \liminf_{m\to\infty} \|\nabla u_{mt}(t)\|_{L^p(t_1,T;L^2(\Omega))}. \end{aligned} \tag{3.22}$$

Since (3.20), we have

$$\begin{aligned} \|\nabla u_m(t)\|^p_{L^p(t_1,T;L^2(\Omega))} &\leq \int_{t_1}^{T} \|\nabla u_m(t)\|^p dt \leq (T-t_1)\left(\frac{2C}{l(t+1)}\right)^{\frac{p}{2}}, \\ \|\nabla u_{mt}(t)\|^p_{L^p(t_1,T;L^2(\Omega))} &\leq \int_{t_1}^{T} \|\nabla u_{mt}(t)\|^p dt \leq (T-t_1)\left(\frac{2C}{\mu(t+1)}\right)^{\frac{p}{2}}. \end{aligned}$$

Then, by (3.22), we obtain

$$\begin{aligned} \|\nabla u\|_{L^p(t_1,T;L^2(\Omega))} &\leq (T-t_1)^{\frac{1}{p}}\left(\frac{2C}{l(t+1)}\right)^{\frac{1}{2}}, \\ \|\nabla u_t\|_{L^p(t_1,T;L^2(\Omega))} &\leq (T-t_1)^{\frac{1}{p}}\left(\frac{2C}{\mu(t+1)}\right)^{\frac{1}{2}}. \end{aligned}$$

So, we deduce

$$\begin{aligned} \|\nabla u\|_{L^\infty(t_1,T;L^2(\Omega))} &= \lim_{p\to\infty} \|\nabla u\|_{L^p(t_1,T;L^2(\Omega))} \leq \left(\frac{2C}{l(t+1)}\right)^{\frac{1}{2}}, \\ \|\nabla u_t\|_{L^\infty(t_1,T;L^2(\Omega))} &= \lim_{p\to\infty} \|\nabla u_t\|_{L^p(t_1,T;L^2(\Omega))} \leq \left(\frac{2C}{\mu(t+1)}\right)^{\frac{1}{2}}. \end{aligned}$$

Since $u, u_t \in C([t_1, T]; H_0^1(\Omega))$ (see Remark 1.2), it follows from the arbitrariness of T and the above inequality that

$$\|\nabla u(t)\|^2 \leq \frac{2C}{l(t+1)} \text{ and } \|\nabla u_t(t)\|^2 \leq \frac{2C}{\mu(t+1)}, \quad t \geq t_1. \tag{3.23}$$

Then the desired result follows from (3.21) and the above inequality.

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