ABOUT THEORY OF PRIMARY DECOMPOSITION OF MONOMIAL IDEAL

Abstract

In this paper, we have to R is a commutative Noetherian ring, i.e. where all ideal is finitely generated, and we have the R-module $I(G)$, which is a monomial ideal, where I(G) is the edge ideal of a simple and finite graph G, with no isolated vertices, which is a finitely generated R-module. We consider also a an ideal of R and N a submodule of $I(G)$ such that $aI(G) \subseteq N$, an inclusion of modules together with the edge ideal. Here in the article, the edge primary decomposition and irreducible decomposition of $a \times N$ are given.

Keywords: monomial ideal, edge primary decomposition, edge dimension filtration, edge ideal of a graph.

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1. Introduction

Throughout this paper, R is a commutative ring with non-zero identity.

We consider $I(G)$, which is an R-module, which is the edge ideal of a graph. In the Cartesian product $R \times I(G)$, define addition and multiplication as follows:

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1).$$

This is called the *edge idealization* of $I(G)$.

In this note, we give an edge primary decomposition of $\mathrm{a} \times N$ via edge primary decompositions of a and N. The edge irreducible decomposition of $\mathrm{a} \times N$ is also presented in the end of the paper.

Associated to the graph G is a monomial ideal

$$I(G) = (v_i v_j \mid v_i v_j \text{ is an edge of } G),$$

with $v_i v_j = v_j v_i$, $i \neq j$, in the polynomial ring $R = K[v_1, v_2, ..., v_s]$ over a field K, called the *edge ideal* of G, for i = 1, …, n, and, j = 1, …, m.

We mean by a graph G, a finite simple graph with the vertex set $V(G)$ having no isolated vertex.

In the Section 2, we presented some prerequisites.

In the Section 3, we presented some results about the theory in question. We can consult the references [1] and [5].

We finalize the paper with a conclusion.

We refer to [3], and [6] for basics in commutative algebra and homological algebra.

2. Prerequisites

This section is in accordance with [2] and [7].

Let $R = K[v_1, ..., v_s]$ be a polynomial ring over a field K, and let $Z = \{z_1, ..., z_q\}$ be a finite set of monomials in R. The *monomial subring* spanned by Z is the K-subalgebra,

$$K[Z] = K[z_1, \ldots, z_q] \subset R.$$

Thus, consider any graph G, simple and finite without isolated vertices, with vertex set $V(G) = \{v_1, \ldots, v_s\}$.

Let Z be the set of all monomials $v_i v_j = v_j v_i$, with $i \neq j$, in $R = K[v_1, \ldots, v_s]$, such that $\{v_i v_j\}$ is an edge of G, i.e., the graph finite and simple G, with no isolated vertices, is such that the squarefree monomials of degree two are defining the edges of the graph G.

If G is a graph without isolated vertices, simple and finite, then let R be denote the polynomial ring on the vertices of G over some fixed field K.

We presented now, the definition of the edge ideal of a graph G, which is finite and simple.

Definition 2.1 [2]**.** According to the previous context, the *edge ideal* of a finite simple graph G, with no isolated vertices, is defined by:

$$I(G) = (v_i v_j \mid v_i v_j \text{ is an edge of } G),$$

with $v_i v_j = v_j v_i$, and also with $i \neq j$,

3. Main Results about Primary Decomposition

Here, we take K a fixed field and we consider $K[v_1, v_2, \ldots, v_s]$, which is the ring polynomial over the field K.

Since K is a field, we have that K is a Noetherian ring and then we have that $K[v_1, \ldots, v_s]$ is also a Noetherian ring (by the Theorem of the Hilbert Basis).

Remark 3.1. By the previous context, $R = K[v_1, v_2, \ldots, v_s]$ is a Noetherian ring. Therefore, the edge ideal $I(G)$ is a finitely generated ideal. Thus, the edge ideal $I(G)$ is a finitely generated R-module, and therefore is a Noetherian R-module.

The terminology and notations of primary decomposition of ideal or submodule, are found in [4]. We put then, the following definition.

Definition 3.2. Let $R = K[v_1, ..., v_s]$. Let N be an R-submodule of $I(G)$. We define the edge primary decomposition $N = N_1 \cap \cdots \cap N_t$ of N being *irredundant* or *minimal* if

(1) the prime ideals $\sqrt{Ann_R(I(G)/N_1)}, ..., \sqrt{Ann_R(I(G)/N_t)}$ are distinct, and

(2) for any $j = 1, ..., t,$ we have $N \neq \bigcap_{i \neq j} N_i$.

Assume, now, that $\mathfrak{a}$ is an ideal of R. In the special case for ideals, an edge primary decomposition $\mathfrak{a} = Q_1 \cap \cdots \cap Q_s$ of $\mathfrak{a}$ is irredundant or minimal if we have $\sqrt{Q_1}, ..., \sqrt{Q_s}$ all distinct, and $\mathfrak{a} \neq \bigcap_{i \neq j} Q_i,$ for any index $j \in \{1, ..., s\}$.

Remark 3.3. We have that $R = K[v_1, ..., v_s]$ is a Noetherian ring, and that $I(G)$ is finitely generated. Then there exists a minimal decomposition of $\mathfrak{a}$ and of N.

And by [5, page 24], we remark that if N is an $\mathfrak{p}$-primary submodule of $I(G)$,

$$Ann_R(I(G)/N) \times I(G),$$

is $\mathfrak{p} \times I(G)$-primary.

Definition 3.4. Let $R = K[v_1, ..., v_s]$. In the context of the Remark 3.3, if N is an $\mathfrak{p}$-primary submodule of $I(G)$ we said to be N an edge $\mathfrak{p}$-primary submodule of $I(G),$ and then we have that

$$Ann_R(I(G)/N) \times I(G),$$

is an edge $\mathfrak{p} \times I(G)$-primary module.

We have now the following result, which we put in the form of a lemma.

Lemma 3.5. [see 7] *Let* $R = K[v_1, \dots, v_s]$. *Let* $\mathfrak{a}$ *be an ideal of R and let N be an R-submodule of* $I(G)$. *Then,* $\mathfrak{a} \times N$ *is edge primary module if and only if either*

(1) $N = I(G)$ *and* $\mathfrak{a}$ *is an edge primary ideal of R or*

(2) $N \subset I(G)$ *and* $N \neq I(G)$, $\mathfrak{a}I(G) \subseteq N$, *and* $\mathfrak{a}$ *and N are edge* $\mathfrak{p}$-*primary where* $\mathfrak{p} = \sqrt{\mathfrak{a}}$.

In either cases, $\mathfrak{a} \times N$ *is an edge* $\sqrt{\mathfrak{a}} \times I(G)$-*primary module.*

Let $\mathfrak{a}$ such that $\mathfrak{a}I(G) \subseteq N$. We presented now more a result. It is useful to get the edge primary decomposition of the ideal $\mathfrak{a} \times N$, which is an *R*-module.

Proposition 3.6. *Let* $R = K[v_1, \dots, v_s]$. *Let* $\mathfrak{a}$ *such that* $\mathfrak{a}I(G) \subseteq N$. *Suppose that* $Ass_R(R/\mathfrak{a}) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$, $Ass_R(I(G)/N) = \{\mathfrak{q}_1, \dots, \mathfrak{q}_t\}$, *and*

$$Ass_R(R/\mathfrak{a}) \bigcap Ass_R(I(G)/N) \neq \varnothing,$$

with $\mathfrak{p}_i = \mathfrak{q}_i$, *for* $i = 1, \dots, r, 1 \leq r \leq \min\{s, t\}$. *Then there exists the following minimal edge primary decompositions of* $\mathfrak{a}$ *and N,*

$$\mathfrak{a} = \bigcap_{i=1}^{s} Q_i; \quad N = \bigcap_{i=1}^{t} N_i$$

such that $Q_i I(G) \subseteq N_i$, *for all* $i = 1, \dots, r$.

Proof. Suppose that

$$\mathfrak{a} = \bigcap_{i=1}^{s} Q_i'; \quad N = \bigcap_{i=1}^{t} N_i',$$

are minimal edge primary decompositions of $\mathfrak{a}$ and *N*. Since

$$\mathfrak{p}_1 = \sqrt{Ann_R(I(G)/N_1')},$$ we have that $N + \mathfrak{p}_1^n I(G) \subseteq N_1'$,

for n large enough.

Set $N_1 = (N + \mathfrak{p}_1^n I(G))_{\mathfrak{p}_1} \cap I(G)$ for n large enough. We have

$$N \subseteq N_1 \subseteq N_1',$$

and N_1 is edge $\mathfrak{p}_1$-primary submodule of $I(G)$. Since

$$N = \bigcap_{i=1}^{t} N_i' \supseteq N_1 \cap \left(\bigcap_{i=2}^{t} N_i' \right) \supseteq N,$$

we have that

$$N = N_1 \cap \left(\bigcap_{i=2}^{t} N_i' \right)$$

is minimal edge primary decomposition of N. Set $Q_1 = (\mathfrak{a} + \mathfrak{p}_1^n) R_{\mathfrak{p}_1} \cap R$.

It is similar as above, we have that

$$\mathfrak{a} = Q_1 \cap \left(\bigcap_{i=1}^{t} Q_i' \right).$$

Note that $Q_1 I(G) \subseteq N_1$.

Moreover, after r steps, set $N_i = N_i'$, for $i = r+1, ..., t$, $Q_i = Q_i'$, for $i = r+1, ..., s$. Thus, we have the requirement. This finishes the proof.

Theorem 3.7. *Let* $R = K[v_1, ..., v_s]$. *Let* $\mathfrak{a}$ *be such that* $\mathfrak{a} I(G) \subseteq N$. *Set*

$$\Lambda_1 = \{i \mid \mathfrak{p}_i \in Ass_R(R/\mathfrak{a}) \cap Ass_R(I(G)/N)\},$$

$$\Lambda_2 = \{i \mid \mathfrak{p}_i \in Ass_R(R/\mathfrak{a}) \backslash Ass_R(I(G)/N)\},$$

$$\Lambda_3 = \{i \mid \mathfrak{q}_i \in Ass_R(I(G)/N) \backslash Ass_R(R/\mathfrak{a})\}.$$

Assume that

$$\mathfrak{a} = \bigcap_{i=1}^{s} Q_i, \quad N = \bigcap_{i=1}^{t} N_i,$$

are minimal edge primary decomposition of $\mathfrak{a}$ *and* N *such that* $Q_i I(G) \subseteq N_i$, *for all* $i \in \Lambda_1$. *Then*

$$\mathfrak{a} \times N = \bigcap_{i\in\Lambda_1} (Q_i \times N_i) \bigcap_{i\in\Lambda_2} (Q_i \times I(G)) \bigcap_{i\in\Lambda_3} (Ann_R(I(G)/N_i) \times N_i)$$

is minimal edge primary decomposition of $\mathfrak{a} \times N$.

Proof. By Lemma 3.5, $Q_i \times N_i$ is edge $\mathfrak{p}_i \times I(G)$-primary module, if $i \in \Lambda_1$, $Q_j \times I(G)$ is edge $\mathfrak{p}_j \times I(G)$-primary module, if $j \in \Lambda_2,$. Also, $Ann_R(I(G)/N_k) \times N_k$ is edge $\mathfrak{p}_k \times I(G)$-primary module, if $k \in \Lambda_3$. Note that $\mathfrak{a} \subseteq Q_i$ for all $i \in \Lambda_1 \bigcup \Lambda_2$. By the hypothesis, it follows that $Q_i I(G) \subseteq N_i$, for all $i \in \Lambda_3$. Then, we have $\mathfrak{a} \subseteq Ann_R(I(G)/N_i)$, for all $i \in \Lambda_3$. On the other hand, we have $N \subseteq N_i$ for all $i \in \Lambda_1 \bigcup \Lambda_3$ and $N \subseteq I(G)$. Thus

$$\mathfrak{a} \times N \subseteq \bigcap_{i\in\Lambda_1} (Q_i \times N_i) \bigcap_{i\in\Lambda_2} (Q_i \times I(G)) \bigcap_{i\in\Lambda_3} (Ann_R(I(G)/N_i) \times N_i).$$

Conversely, assume that

$$(a,\ x) \in \bigcap_{i\in\Lambda_1} (Q_i \times N_i) \bigcap_{i\in\Lambda_2} (Q_i \times I(G)) \bigcap_{i\in\Lambda_3} (Ann_R(I(G)/N_i) \times N_i).$$

Then, we have

$$a \in \bigcap_{i\in\Lambda_1\bigcup\Lambda_2} Q_i \bigcap_{i\in\Lambda_3} Ann_R(I(G)/N_i) \subseteq \bigcap_{i=1}^{s} Q_i = \mathfrak{a}.$$

On the other hand,

$$x \in \bigcap_{i \in \Lambda_1 \cup \Lambda_3} N_i \cap I(G) = \bigcap_{i=1}^{t} N_i = N.$$

This prove that $(a, x) \in \mathfrak{a} \times N$. Thus, it follows that the edge primary decompositions of $\mathfrak{a} \times N$ is minimal. This finishes the proof.

From now on, we assume that $\dim(I(G)) = t$.

Definition 3.8. Let $R = K[v_1, ..., v_s]$. For an integer $0 \le i < t$, let $I(G)_i$ denote the largest submodule of $I(G)$ such that $\dim_R(I(G)_i) \le i$, and $I(G)_t = I(G)$. Since $I(G)$ is a Noetherian R-module, we have that there exists $I(G)_i$ and also, $I(G)_{i-1} \subseteq I(G)$.

The increasing filtration $\{I(G)_i\}_{0 \le i \le t}$ of submodules of $I(G)$ is called *edge dimension filtration* of $I(G)$.

Lemma 3.9. [see 1] *Let* $R = K[v_1, ..., v_s]$. *Assume that*

$$0 = \bigcap_{i=1}^{t} N_i$$

is a minimal edge primary decomposition of 0 *in* $I(G)$. *Then*

$$I(G)_i = \bigcap_{\dim(R/\mathfrak{p}_j) > i} N_j.$$

The following result give a description of edge dimension filtration of edge idealization of $I(G)$.

Theorem 3.10. *Let* $R = K[v_1, ..., v_s]$.

Assume that $\{R_i\}_{i=0,...,d}$ *and* $\{I(G)_i\}_{i=0,...,t}$ *are edge dimension filtration of R and* $I(G)$, *respectively. Set* $W = R \times I(G)$, $W_i = R_i \times I(G)_i$, *for* $i = 0, ..., t$, *and* $W_i = R_i \times I(G)$, *for* $i = t+1, ..., d$. *Then, we have* *that* $\{W_i\}_{i=0,\dots,d}$ *is the edge dimension filtration of W.*

Proof. Set $\Lambda_i^k = \{i \in \Lambda_i \,|\, \dim((R \times I(G))/(\mathfrak{p}_i \times I(G))) > k\}$, for $i = 1, 2, 3.$ Thus, we have that

$$(R \times I(G))/(Q_i \times N_i) \cong R/Q_i \times I(G)/N_i$$

and

$$\dim((R \times I(G))/(Q_i \times N_i)) = \dim(R/Q_i) = \dim(I(G)/N_i)$$

for all $i \in \Lambda_1$. Moreover, we have that

$$(R \times I(G))/(Q_i \times I(G)) \cong R/Q_i \times I(G)$$

and

$$\dim((R \times I(G))/(Q_i \times I(G))) = \dim(R/Q_i)$$

for all $i \in \Lambda_2$. Also, we have that

$$(R \times I(G))/(Ann_R(I(G)/N_i) \times N_i) \cong R/Ann_R(I(G)/N_i) \times I(G)/N_i$$

and

$$\dim((R \times I(G))/(Ann_R(I(G)/N_i) \times N_i))$$

$$= \dim(R/Ann_R(I(G)/N_i)) = \dim(I(G)/N_i)$$

for all $i \in \Lambda_3$. So, by Lemma 3.9,

$$W_k = \bigcap_{i \in \Lambda_2^k} (Q_i \times I(G)) = \left(\bigcap_{\dim(R/\mathfrak{p}_i) > k} Q_i \right) \times I(G) = R_k \times I(G)$$

if $k = t+1, \dots, d$ and

$$W_k = \bigcap_{i \in \Lambda_1^k} (Q_i \times N_i) \bigcap_{i \in \Lambda_2^k} (Q_i \times I(G)) \bigcap_{i \in \Lambda_3^k} (Ann_R(I(G)/N_i) \times N_i)$$

$$= \left(\bigcap_{\dim(R/\mathfrak{p}_i)>k} Q_i \right) \times \left(\bigcap_{\dim(R/\mathfrak{p}_i)>k} N_i \right) = R_i \times I(G)_i$$

if $k = 0, \ldots, t.$ This completes the proof.

Lemma 3.11. *Let* $R = K[v_1, \ldots, v_s].$ *Let* $\mathfrak{a}$ *be such that* $\mathfrak{a}I(G) \subseteq N.$ *Then* $\mathfrak{a} \times N$ *is edge irreducible if and only if either*

(1) $N = I(G)$ *and* $\mathfrak{a}$ *is an irreducible ideal of R, or*

(2) $N \subseteq I(G),$ *and* $N \neq I(G),$ $\mathfrak{a} = Ann_R(I(G)/N),$ *and* N *is irreducible.*

Proof. (1) It follows by noting that every ideal of $R \times I(G)$ which contains $\mathfrak{a} \times I(G)$ has the form $J \times I(G),$ with $\mathfrak{a} \subseteq J.$

(2) It follows from [1, Proposition 4.4]. This finishes the proof.

Theorem 3.12. *Let* $R = K[v_1, \ldots, v_s].$ *Let* $\mathfrak{a}$ *be such that* $\mathfrak{a}I(G) \subset N.$

Assume that

$$\mathfrak{a} = \bigcap_{i=1}^{t} Q_i; \quad N = \bigcap_{i=1}^{r} N_i$$

are edge irreducible decomposition of $\mathfrak{a}$ *and N. Then*

$$\mathfrak{a} \times N = \bigcap_{i=1}^{t} (Q_i \times I(G)) \bigcap_{i=1}^{r} (Ann_R(I(G)/N_i) \times N_i)$$

is edge irreducible decomposition of $\mathfrak{a} \times N.$

Proof. By Lemma 3.11, we have that $Q_i \times I(G),$ is edge irreducible, for $i = 1, \ldots, t,$ and also we have that $Ann_R(I(G)/N_i) \times N_i,$ for $i = 1, \ldots, r,$ is edge irreducible. Thus, we have the requirement.

Conclusion

With the results of this paper, we introduce commutative algebra theory as a useful tool for application in graph theory. And so we provide a relation of the general theory of modules in a particular case, making the necessary considerations, to obtain relevant results within the study area in question. We can also find some things from this theory in [8], [9], [10] and [11].

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