DATA DEPENDENCE OF RATIONAL-TYPE CONTRACTIVE CONDITIONS IN S-METRIC SPACE

Abstract

In this research work, we investigate the data dependence of the fixed point in S-metric space for some general rational type contractive conditions. To ascertain this, the existence and uniqueness of the fixed point were first determined. Our results will extend some know results in the literature.

Keywords: `Data dependence`, Rational-type contraction, Existence and uniqueness, S-metric space

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1. Introduction

Among the primary instruments used in the study of fixed point theory are contractive mappings and iteration processes. In the literature of this very active field of study, numerous writers have created and developed iteration systems and contractive mappings for a variety of uses. It is crucial to determine whether an iteration technique utilized in any study converges to a fixed point of a contractive type mapping that corresponds to a specific iteration process. As a result, it makes sense that there are a lot of works about the convergence of iteration techniques. Investigating a wide range of topics, including the existence and uniqueness of fixed points, their construction, etc., is the focus of fixed point theory.

[1] addressed the challenge of examining the continuous dependence of the fixed points in normed linear space for both Schaefer and Mann iteration processes using a (φ, ψ)-contractive condition in order to provide some answers to the question posed by [2] that, aside from the Picard iteration process, the continuous dependence of the fixed points has not been studied thus far for other fixed point iteration procedures. There results are new extensions of some of the results of [2].

[3] established conditions for a continuous dependence of fixed points and an application to non-linear functional differential equation of neutral type. [4] investigated the continuous dependence of the fixed points in uniformly convex Banach space for nonexpansive and quasi-nonexpansive mappings. Their results extended some recently announced ones in the current literature.

[5]Several findings on continuous dependence for implicit Kirk-Mann and implicit Kirk-Ishikawa iterations have been thoroughly examined. They created a general hyperbolic space, which is undoubtedly a free associate of several existing hyperbolic spaces, in order to equipoise the development of these algorithms. The discoveries that were presented were extensions of previous findings and have a wide range of potential uses. For more work on data dependence (see [6], [7], [8], [9]).

In this work, we study the data dependence of a fixed point using some general rational type contractive condition defined in the setting of S-metric space.

2. Preliminary

Definition 2.1 [10]: Let X be a set. A function $S : X^3 \to [0, \infty)$ is said to be a S-metric spaces on a nonempty set X, for all $x, y, z, a \in X$, and the following conditions hold:

1. $S(x, y, z) \geq 0$
2. $S(x, y, z) = 0$ if and only if $x = y = z$
3. $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z, a \in X$ (rectangle inequality)

Hence, the function S is called an S-metric on X and the pair (X, S) is called an S- metric space.
Example 2.1 [10]: Let $X = R^n$ and $||.||$ a norm on X, then

$$S(x, y, z) = ||y + z - 2x|| + ||y - z||$$

is an S-metric on X.
Let X be a nonempty set, d is ordinary metric on X, then $S(x, y, z) = d(x, z) + d(y, z)$ is an S-metric on X. This S-metric is called the usual S-metric on X.

Definition 2.2 [10]: Let (X, S) be an S-metric space.

1. A sequence $\{x_n\} \in X$ converges to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to +\infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in N$ such that for all $n \geq n_0$ we have $S(x_n, x_n, x_m) < \epsilon$. We write $x_n \to x$ for brevity;
2. A sequence$\{x_n\} \in X$ is a Cauchy sequence if $S(x_n, x_n x_m) \to 0$ as $n \to +\infty$.
   That is, for each $\epsilon \geq 0$, there exists $n_0 \in N$ such that for all $n \geq n_0$ we have $S(x_n, x_n, x_m) < \epsilon$; and
3. The S-metric space (X, S) is complete if every Cauchy sequence is a convergent sequence.

Lemma 2.1 [10]: In an S-metric space, we have $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.
Lemma 2.2 [10] Let (X, S) be an S-metric space. If $x_n \to x$ and $y_n \to y$ then $S(x_n, x_n, y_n) \to S(x, x, y)$.

Example 2.2 [10] Let $\mathbb{R}$ be the real line. Then

$$S(x, y, z) = |x - z| + |y - z|$$

for all $x, y, z \in \mathbb{R}$ is an S-metric on $\mathbb{R}$. This S-metric on $\mathbb{R}$ is called the usual S-metric on $\mathbb{R}$.

Definition 2.3 [11] : There exist some nonnegative real number α, β *and* γ such that

$$S(Tx, Tx, Ty) \leq \alpha \frac{S(x, x, Tx)S(y, y, Ty)}{S(x, x, y)} + \beta[S(x, x, Tx) + S(y, y, Ty)] + \gamma S(x, x, y)$$
$$+ L\ min[S(x, x, Ty), S(y, y, Tx)]$$

for all $x \neq y \in X$ with $x \leq y$ where $L \geq 0$ and $\alpha, \beta, \gamma \in [0, 1)$ with $0 \leq \alpha + 2\beta + \gamma < 1$.

Definition 2.4 [11] : There exist some nonnegative real number α, β such that

$$S(Tx, Tx, Ty) \leq \alpha \frac{S(y, y, Ty)(1 + S(y, y, Ty))}{1 + S(x, x, y)} + \beta S(x, x, y)$$

where T is continuous and $\alpha + \beta < 1$

Definition 2.5 [12]: A self mapping $T : X \to X$ is said to be (α, Ψ) rational type-I contraction if there exists a function $\psi \in \Psi$, such that for all $x, y \in X$ the following conditions holds.

$$\alpha(x, x, y)S(Tx, Tx, Ty) \leq \psi(M(x, x, y))$$

where

$$M(x, x, y) = max[S(x, x, y), S(x, x, Tx), S(y, y, Ty),$$
$$\frac{S(x, x, Tx)S(y, y, Ty)}{1 + S(x, x, y)}, \frac{S(x, x, Tx)S(y, y, Ty)}{1 + S(Tx, Tx, y)}]$$

Definition 2.6 [11] : Let (X, S) be an S-metric space. A self-mapping T on X is called an almost Jaggi contraction if it satisfies the following condition.

$$S(Tx, Ty, Tz) \leq \alpha \frac{S(x, x, Tx).S(y, y, Ty)}{S(x, y, z)} + \beta S(x, y, z)$$
$$+ L\min\{S(x, x, Ty), S(y, y, Tx)\}$$

Definition 2.7 [11]: Let us consider $S \neq \phi$, and the map. $\alpha : X \times X \times X \to [0, \infty)$. An operator $T : X \to X$ is α-admissible if $\alpha(x, y, z) \geq 1$ imply that $\alpha(Tx, Ty, Tz) \geq 1$, for all $x, y, z \in S$

Definition 2.8 [11]: Let (X, S) be a Jleli-Samet space and$\alpha : X \times X$ is called JS α-regular if for $\{x_n\}$ convergent to x and $\alpha(x_n, x_{n+1}) \geq 1$, there is a subsequence of the initial sequence such that $\alpha(x_n, x_n, x) \geq 1$, for all $n \in \mathbb{N}$

Let Ψ be the family of functions $\psi : [0, \infty) \to [0, \infty)$ such that the following properties are accomplished:

i ψ is upper semi continuous and strictly increase;

ii $\{\psi^n\}$ converges to 0, for all $t > 0$

3. Main Result

Theorem 3.1. *Let (X, S) be a complete S-metric space, $T : X \to X$ be an operator and $\alpha : X^3 \to [0, \infty)$ be a given mapping. Suppose that the following conditions are satisfied:*

(a) T is an α-admissible mapping;

(b) there is $x_0 \in X$ with $\delta(S, T, x_0) < \infty$ such that $\alpha(x_0, Tx_0, Tx_0) \geq 1$;

(c) there is a function $\psi \in \Psi$ such that $\alpha(x, y, z)S(Tx, Ty, Tz) \leq \psi(M(x, y, z))$ where,

$$M(x,y,z) = max[S(x,y,z), S(x,x,Tx), \frac{S(x,x,Tx)S(y,y,Ty)}{1+S(x,y,z)}, \frac{S(x,x,Tx)S(y,y,Ty)}{1+S(Tx,Tx,Ty)}] \tag{1}$$

for all $x, y, z \in X$

Then the sequence Tx_n has a unique fixed point and it is also a data dependence.

Proof. Let $x_0 \in X$ and denote $x_n = T^n x_0$ the Picard sequence where $n \in \mathbb{N}$. If we have $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, the x_n is the fixed point of T.
Let consider the case where $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$.
Then, we will show that $\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0$ by taking the advantage of the properties of the mapping α, ψ and that of S-metric space. Knowing that T is an α-admissible mapping i.e
$\alpha(x_0, x_0, Tx_0) \geq 1$ implies $\alpha(Tx_0, Tx_0, Tx_1) = \alpha(x_1, x_1, x_2) \geq 1$
From the contraction relation we have

$$\begin{aligned} S(x_{n+1}, x_{n+1}, x_{n+2}) &= S(Tx_n, Tx_n, Tx_{n+1}) \\ &\leq \alpha(x_n, x_n, x_{n+1})S(Tx_n, Tx_n, Tx_{n+1}) \\ &\leq \psi(M(x_n, x_n, x_{n+1})) \end{aligned} \tag{2}$$

Given that,

$$\begin{aligned} M(x_n, x_n, x_{n+1}) &= \max[S(x_n, x_n, x_{n+1}), S(x_n, x_n, Tx_n), \\ &\frac{S(x_n, x_n, Tx_n)S(x_{n+1}, x_{n+1}, Tx_{n+1})}{1 + S(x_n, x_n, x_{n+1})}, \\ &\frac{S(x_n, x_n, Tx_n)S(x_{n+1}, x_{n+1}, Tx_{n+1})}{1 + S(Tx_n, Tx_n, Tx_{n+1})}] \qquad (3) \\ &= \max\left[S(x_n, x_n, x_{n+1}), \frac{S(x_n, x_n, Tx_n)S(x_{n+1}, x_{n+1}, Tx_{n+1})}{1 + S(Tx_n, Tx_n, Tx_{n+1})}\right] \qquad (4) \\ &= S(x_n, x_n, x_{n+1}) \qquad (5) \end{aligned}$$

therefore,

$$
\begin{aligned}
S(x_{n+1}, x_{n+1}, x_{n+2}) &\leq \psi S(x_n, x_n, x_{n+1}) \\
&= \psi S(Tx_{n-1}, Tx_{n-1}, Tx_n) \\
&\leq \psi^2(x_{n-1}, x_{n-1}, x_n) \\
&= \psi^2(Tx_{n-2}, Tx_{n-2}, Tx_{n-1}) \\
&\leq \psi^3(x_{n-2}, x_{n-2}, x_{n-1}) \\
&. \\
&. \\
&. \\
&\leq \psi^{n+1}(x_0, x_0, x_1) \qquad (6)
\end{aligned}
$$

which shows the sequence $\{x_{n+1}\}$ converges.

Next is to show the sequence is also a Cauchy sequence

$$
\begin{aligned}
S(x_n, x_n, x_m) &\leq S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_{n+2}) + S(x_{n+2}, x_{n+2}, x_{n+3}) \\
&+ ...S(x_{m-1}, x_{m-1}, x_m) \qquad (7) \\
&\leq \psi^n S(x_0, x_0, x_1) + \psi^{n+1} S(x_0, x_0, x_1) + \psi^{n+2} S(x_0, x_0, x_1) \\
&+ ...\psi^{m-1} S(x_0, x_0, x_1) \\
&= [\psi^n + \psi^{n+1}\psi^{n+2} + ...\psi^{m-1}] S(x_0, x_0, x_1) \\
&\leq \frac{\psi^n}{1-\psi} S(x_0, x_0, x_1) \qquad (8)
\end{aligned}
$$

$$S(x_n, x_n, x_m) \leq S(x_0, x_0, x_1)$$

which prove the sequence $\{x_{n+1}\}$ is an S Cauchy sequence and is S-complete.

If the limit of the sequence is k then, $Tk = k$

Therefore,

$$
\begin{aligned}
S(x_{n+1}, x_{n+1}, k) &= S(Tx_n, Tx_n, Tk) \\
&\leq \alpha(x_n, x_n, k) S(Tx_n, Tx_n, Tk) \\
&\leq \psi(M(x_n, x_n, k)) \qquad (9)
\end{aligned}
$$

Given that,

$$\begin{aligned} M(x_n, x_n, k) =&\max[S(x_n, x_n, k), S(x_n, x_n, k), \\ &\frac{S(x_n, x_n, Tx)S(k, k, Tk)}{1 + S(x_n, x_n, k)}, \\ &\frac{S(x_n, x_n, Tx_n)S(k, k, Tk)}{1 + S(Tx_n, Tx_n, Tk)}] \\ &= S(x_n, x_n, k) \end{aligned} \tag{10}$$

Combining the properties of the sequence, if the exists $N \in \mathbb{N}$ such that $S(x_n, x_n, k) < \epsilon$, $S(x_n, x_n, x_{n+1}) < \epsilon$ for all $n \geq \mathbb{N}$ where $\epsilon > 0$ is arbitrary chosen. It follows that $\lim_{n\to\infty} S(x_{n+1}, x_{n+1}, Tk) = 0$ hence, $x_{n+1} = Tk = k$.

Suppose, q is another fixed point of T endowed with the above mentioned properties.

Then,

$$\begin{aligned} S(k, k, q) =&S(Tk, Tk, Tq) \\ &\leq \alpha(k, k, q)S(Tk, Tk, Tq) \\ &\leq \psi(M(k, k, q)) \end{aligned} \tag{11}$$

where,

$$\begin{aligned} M(k, k, q) =&\max[S(k, k, q), S(k, k, Tk), \\ &\frac{S(k, k, Tk)S(q, q, Tq)}{1 + S(k, k, q)}, \\ &\frac{S(k, k, Tk)S(q, q, Tq)}{1 + S(Txk, Tk, Tq)}] \\ &= S(k, k, q) \end{aligned} \tag{12}$$

$$S(k, k, q) = \psi S(k, k, q) < S(k, k, q) \tag{13}$$

which is a contradiction.

Hence,

$$k = q \tag{14}$$

From the above equation, we realized that the sequence $\{x_n\}$ has a unique fixed point.

Next, is to show that it also a data dependence.

Suppose $U : X \to X$ also satisfying condition (a)-(c) which is an approximate operator of T, then U will have a fixed point say x^*, if there exists $\eta > 0$ such that

$$S(Tx, Tx, Ux) \leq \eta \tag{16}$$

then

$$S(k,k,x^*) \leq t_\eta \tag{17}$$

where

$$F_T = \{k\}$$

the fixed point of T and U are k and x^* respectively.
Therefore,

$$\begin{aligned} S(k,k,x^*) =& S(Tk,Tk,Ux^*) \\ & \leq S(Tk,Tk,Tx^*) + S(Tx^*,Tx^*,Ux^*) & (18) \\ & \leq S(Tk,Tk,Tx^*) + \eta \\ & \leq \alpha(k,k,x^*)S(Tk,Tk,Tx^*) + \eta \\ & \leq \psi(M(k,k,x^*)) + \eta & (19) \end{aligned}$$

where

$$\begin{aligned} M(k,k,x^*) =& max[S(k,k,x^*), S(k,k,x^*), \frac{S(k,k,x^*)S(x^*,x^*,Tx^*)}{1+S(k,k,x^*)}, \\ & \frac{S(k,k,Tk)S(x^*,x^*,Tx^*)}{1+S(Tk,Tk,Tx^*)}] \\ & = S(k,k,x^*) & (20) \end{aligned}$$

$$\begin{aligned} S(k,k,x^*) \leq& \psi S(k,k,x^*) + \eta \\ & S(k,k,x^*) - \psi S(k,k,x^*) \leq \eta & (21) \\ & (1-\psi)S(k,k,x^*) \leq \eta \\ & S(k,k,x^*) \leq \frac{\eta}{1-\psi} \\ & S(k,k,x^*) \leq t_\eta & (22) \end{aligned}$$

since U is an approximate operator of T, then $\psi S(k,k,x^*) \to 0$ as $\eta \to 0$ □

Theorem 3.2. *Let (X,S) be a complete S-metric space. Suppose that a self-mapping defined as*

$$\begin{aligned} S(Tx,Ty,Tz) \leq& \alpha \frac{S(x,x,Tx).S(y,y,Ty)}{S(x,y,z)} + \beta S(x,y,z) \\ & + Lmin\{S(x,x,Ty), S(y,y,Tx)\} & (23) \end{aligned}$$

for any distinct $x,y,z \in X$ with $x \leq y$, where $L > 0$ and $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$ is an almost Jaggi contraction, continuous and non-decreasing. Suppose there exists $x_0 \in X$ with $x_0 \leq Tx_0$. Then T has a unique fixed point and also a data dependence.

Proof. Let $x_0 \in X$ and set $x_{n+1} = Tx_n$. Let $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$ since $x_0 \leq Tx_0$, then $x_0 \leq x_1 \leq x_{\leq} ... \leq x_n \leq x_{n+1} \leq$

Now,

$$\begin{aligned} S(x_n, x_n, x_{n+1}) =& S(Tx_{n-1}, Tx_{n-1}, Tx_n) \qquad (24) \\ & \alpha \frac{S(x_{n-1}, x_{n-1}, Tx_{n-1}) S(x_n, x_n, Tx_n)}{S(x_{n-1}, x_{n-1} x_n)} \\ & + \beta S(x_{n-1}, x_{n-1}, x_n) + \\ & L\min[S(x_{n-1}, x_{n-1}, Tx_n), S(x_n, x_n, Tx_{n-1})] \end{aligned}$$

(25)

$$\begin{aligned} S(x_n, x_n, x_{n-1}) - \alpha S(x_n, x_n, x_{n-1}) &\leq (\beta + L) S(x_{n-1}, x_{n-1}, x_n) \\ (1-\alpha) S(x_n, x_n, x_{n+1}) &\leq (\beta + L) S(x_{n-1}, x_{n-1}, x_n) \\ S(x_n, x_n, x_{n+1}) &\leq \frac{(\beta + L)}{1-\alpha} S(x_{n-1}, x_{n-1}, x_n) \\ S(x_n, x_n, x_{n+1}) &\leq k S(x_{n-1}, x_{n-1}, x_n) \qquad (26) \end{aligned}$$

where

$$k = \frac{(\beta + L)}{1-\alpha}$$

$$\begin{aligned} S(x_n, x_n, x_{n-1}) =& kS(Tx_{n-2}, Tx_{n-2}, Tx_{n-1}) \\ & \leq k^2 S(x_{n-2}, x_{n-2}, x_{n-1}) \\ & = k^2 S(Tx_{n-3}, Tx_{n-3}, Tx_{n-2}) \\ & \leq k^3 S(x_{n-3}, x_{n-3}, x_{n-2}) \\ & . \\ & . \\ & . \\ & \leq k^n S(x_0, x_0, x_1) \qquad (27) \end{aligned}$$

which prove the sequence converges.

Next, is to show the sequence is a Cauchy sequence.

$$\begin{aligned} S(x_n, x_n, x_m) \leq & S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_{n+2}) \\ & + S(x_{n+2}, x_{n+2}, x_{n+3}) + ... + S(x_{m-1}, x_{m-1}, x_m) \\ & \leq k^n S(x_0, x_0, x_1) + k^{n+1} S(x_0, x_0, x_1) \\ & + k^{n+2} S(x_0, x_0, x_1) + ... + k^{m-1} S(x_0, x_0, x_1) \\ & \leq [k^{n+1} + k^{n+2} + ... + k^{m-1}] S(x_0, x_0, x_1) \\ & \leq \frac{k^n}{1-k} S(x_0, x_0, x_1) \\ & \leq S(x_0, x_0, x_1) \end{aligned} \tag{28, 29}$$

This clearly show the sequence is a S-Cauchy sequence and since the S-Cauchy sequence converges then the sequence is also S- complete.

Since the sequence converges then it has a limit, if the limit of the sequence is x^* then $Tx^* = x^*$

Next, we show the limit is the fixed point of the sequence.

$$\begin{aligned} S(x_{n+1}, x_{n+1}, x^*) = & S(Tx_n, Tx_n, Tx^*) \\ & \alpha \frac{S(x_n, x_n, Tx_n) S(x^*, x^*, Tx^*)}{S(x_n, x_n, x^*)} \\ & + \beta S(x_n, x_n, x^*) + \\ & L\min[S(x_n, x_n, Tx^*), S(x^*, x^*, Tx_n)] \\ & \leq \beta S(x_n, x_n, x^*) + LS(x_n, x_n, Tx^*) \\ & = (\beta + L) S(x_n, x_n, x^*) \end{aligned} \tag{30, 31}$$

$$S(x_{n+1}, x_{n+1}, x^*) \leq S(x_n, x_n, x^*) \tag{32}$$

since $\beta + L < 1$

limit at $n \to \infty$

$$S(x_{n+1}, x_{n+1}, x^*) = 0$$

hence $x_{n+1} \to x^*$

Suppose y^* is another fixed point of T then,

$$
\begin{aligned}
S(x^*,x^*,y^*) =& S(Tx^*,Tx^*,Ty^*) \\
& \alpha\frac{S(x^*,x^*,Tx^*)S(y^*,y^*,Ty^*)}{S(x^*,x^*,y^*)} \\
& +\beta S(x^*,x^*,y^*)+ \\
& L\min[S(x^*,x^*,Ty^*),S(y^*,y^*,Tx^*)] \\
& \leq \beta S(x^*,x^*,y^*)+LS(x^*,x^*,Ty^*) \\
& \leq [\beta+L]S(x^*,x^*,y^*) < S(x^*,x^*,y^*)
\end{aligned} \tag{33}
$$

which is a contradiction, hence $x^* = y^*$. Hence the fixed is unique.

Suppose $U : X \to T$ is an approximate operator of T, then U will also have fixed point say q^*, if there exists $\eta > 0$ such that

$$S(Tx,Tx,Ux) \leq \eta$$

then

$$S(x^*,x^*,q^*) \leq t_\eta$$

where,$F_T = \{x^*\} \quad F_U = \{q^*\}$

$$
\begin{aligned}
S(x^*,x^*,q^*) =& S(Tx^*,Tx^*,Uq^*) \\
& \leq S(Tx^*,Tx^*,Tq^*)+S(Tq^*,Tq^*,Uq^*) \\
& \leq S(Tx^*,Tx^*,Tq^*)+\eta \\
& \leq \alpha\frac{S(x^*,x^*,Tx^*)S(q^*,q^*,Tq^*)}{S(x^*,x^*,q^*)} \\
& +\beta S(x^*,x^*,q^*)+ \\
& L\min[S(x^*,x^*,Tq^*),S(q^*,q^*,Tx^*)]+\eta \\
& \leq \beta S(x^*,x^*,q^*)+LS(x^*,x^*,Tq^*)+\eta \qquad (35) \\
& \leq [\beta+L]S(x^*,x^*,q^*)+\eta
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
S(x^*,x^*,q^*)-[\beta+L]S(x^*,x^*,q^*) &\leq \eta \\
[1-\beta-L]S(x^*,x^*,q^*) &\leq \eta \\
S(x^*,x^*,q^*) &\leq \frac{\eta}{1-\beta-L} \\
S(x^*,x^*,q^*) &\leq t_\eta
\end{aligned} \tag{36}
$$

The convergence of the fixed points F_T and F_U depends on how fast $\eta \to 0$. Hence, T and U are data dependence on η. □

Theorem 3.3. *Let (X,S) be a complete S-metric space, suppose a self-mapping T on X is continuous, non-decreasing and satisfies the contraction condition*

$$S(Tx,Ty,Tz) \leq \alpha\frac{S(x,x,Tx)S(y,y,Ty)}{S(x,x,y)} + \beta[S(x,x,Tx)+S(y,y,Ty)+\gamma S(x,x,y)] + L\min[S(x,x,Ty),S(y,y,Tx)] \tag{37}$$

for all $x \neq Y \in$ with $x \leq y$ where $l \geq 0$ and $\alpha,\beta,\gamma \in [0,1)$ with $0 \leq \alpha+2\beta+\gamma < 1$ then T has a fixed point and also a data dependence.

Proof. Let $x_0 \in X$ and set $x_{n+1} = Tx_n$. Let $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$ since $x_0 \leq Tx_0$,
Now,

$$\begin{aligned} S(x_n,x_n,x_{n+1}) =& S(Tx_{n-1},Tx_{n-1},Tx_n) & (37)\\ &\leq \alpha\frac{S(x_{n-1},x_{n-1},Tx_{n-1})S(x_n,x_n,Tx_n)}{S(x_{n-1},x_{n-1},x_n)} \\ &+ \beta[S(x_{n-1},x_{n-1},Tx_{n-1})+S(x_n,x_n,Tx_n) \\ &+ \gamma S(x_{n-1},x_{n-1},x_n)] \\ &+ L\min[S(x_{n-1},x_{n-1},Tx_n),S(x_n,x_n,Tx_{n-1})] \\ &= \alpha\frac{S(x_{n-1},x_{n-1},x_n)S(x_n,x_n,Tx_n)}{S(x_{n-1},x_{n-1},x_n)} & (38)\\ &+ \beta[S(x_{n-1},x_{n-1},x_n)+S(x_n,x_n,x_{n+1}) \\ &+ S(x_n,x_n,x_{n+1})+\gamma S(x_{n-1},x_{n-1},x_n)] \\ &+ L\min[S(x_{n-1},x_{n-1},x_{n+1}),S(x_n,x_n,x_n)] \\ &\leq \alpha S(x_n,x_n,x_{n+1})+\beta[S(x_{n-1},x_{n-1},x_n)+S(x_n,x_n,x_{n+1}) \\ &+ S(x_n,x_n,x_{n+1})+\gamma S(x_{n-1},x_{n-1},x_n)] \\ &+ L\min[(S(x_{n-1},x_{n-1},x_n)+S(x_n,x_n,x_{n+1})),S(x_n,x_n,x_n)] & (39) \end{aligned}$$

$$\begin{aligned} S(x_n,x_n,x_{n+1}) \leq& \alpha S(x_n,x_n,x_{n+1})+\beta[S(x_{n-1},x_{n-1},x_n) \\ &+ 2S(x_n,x_n,x_{n+1})+\gamma S(x_{n-1},x_{n-1},x_n)] \\ &\leq \alpha S(x_n,x_n,x_{n+1})+\beta[2S(x_n,x_n,x_{n+1}) \\ &+ (1+\gamma)S(S(x_{n-1},x_{n-1},x_n))] \\ &\leq S(x_n,x_n,x_{n+1})+2\beta S(x_n,x_n,x_{n+1}) \\ &+ \beta(1+\gamma)S(x_{n-1},x_{n-1},x_n) \end{aligned}$$

$$S(x_n, x_n, x_{n+1}) - (\alpha + 2\beta)S(x_n, x_n, x_{n+1}) \leq \beta(1+\gamma)S(x_{n-1}, x_{n-1}, x_n) \tag{39}$$

$$(1 - \alpha - 2\beta)S(x_n, x_n, x_{n+1}) \leq \beta(1+\gamma)S(x_{n-1}, x_{n-1}, x_n)$$

$$S(x_n, x_n, x_{n+1}) \leq \frac{\beta(1+\gamma)}{(1-\alpha-2\beta)}S(x_{n-1}, x_{n-1}, x_n)$$

$$let \quad k = \frac{\beta(1+\gamma)}{(1-\alpha-2\beta)} < 1$$

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &\leq kS(x_{n-1}, x_{n-1}, x_n) \\ &= kS(Tx_{n-2}, Tx_{n-2}, Tx_{n-1}) \\ &\leq k^2kS(x_{n-2}, x_{n-2}, x_{n-1}) \\ &= k^2kS(Tx_{n-3}, Tx_{n-3}, Tx_{n-2}) \\ &\leq k^3S(x_{n-3}, x_{n-3}, x_{n-2}) \\ &. \\ &. \\ &. \end{aligned}$$

$$S(x_n, x_n, x_{n+1}) \leq k^n S(x_0, x_0, x_1) \tag{40}$$

which prove the sequence is S-convergent

Next, is to prove the sequence is S-Cauchy.

Therefore,

$$\begin{aligned} S(x_n, x_n, x_m) \leq & S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_{n+2}) \qquad (41) \\ & + S(x_{n+2}, x_{n+2}, x_{n+3}) + ... + S(x_{m-1}, x_{m-1}, x_m) \\ & \leq k^n S(x_0, x_0, x_1) + k^{n+1}S(x_0, x_0, x_1) \\ & + k^{n+2}S(x_0, x_0, x_1) + ... + k^{m-1}S(x_0, x_0, x_1) \\ & \leq [k^{n+1} + k^{n+2} + ... + k^{m-1}]S(x_0, x_0, x_1) \\ & \leq \frac{k^n}{1-k}S(x_0, x_0, x_1) \\ & \leq S(x_0, x_0, x_1) \qquad (42) \end{aligned}$$

$$S(x_{n+1}, x_{n+1}, x^*) \leq S(x_n, x_n, x^*) \tag{43}$$

Then the sequence is S-Cauchy and by the convergence of the Cauchy sequence then the sequence is S-complete.

Since the sequence converges it implies it has a limit.

Let the limit of the sequence be x^* then we $Tx^* = x^*$ is the fixed point of the sequence

Hence,

$$
\begin{aligned}
S(x_{n+1}, x_{n+1}, x^*) =& S(Tx_n, Tx_n, Tx^*) \qquad (44) \\
&\leq \alpha \frac{S(x_n, x_n, Tx_n) S(x^*, x^*, Tx^*)}{S(x_n, x_n, x^*)} \\
&+ \beta[S(x_n, x_n, Tx_n) + S(x^*, x^*, Tx^*) \\
&+ \gamma S(x_n, x_n, x^*)] \\
&+ L \min[S(x_n, x_n, Tx^*), S(x^*, x^*, Tx_n)] \\
&\leq \beta[S(x_n, x_n, Tx_n) + \gamma S(x_n, x_n, x^*)] \\
&+ L \min[S(x_n, x_n, Tx^*)] \\
&\leq \beta S(x_n, x_n, x_{n+1}) + [\gamma + L] S(x_n, x_n, x^*) \\
&\leq S(x_n, x_n, x^*) + S(x^*, x^*, x_{n+1}) \\
&+ [\gamma + L] S(x_n, x_n, x^*) \\
&\leq [\beta + \gamma + L] S(x_n, x_n, x^*) + \beta S(x^*, x^*, x_{n+1}) \\
&\leq \frac{[\beta + \gamma + L]}{1 - \beta} S(x_n, x_n, x^*) \qquad (45)
\end{aligned}
$$

Taking limit at $n \to \infty$ and for $S(x_{n+1}, x_{n+1}, x^*) < \epsilon$, $S(x_n, x_n, x^*) < \epsilon$, $S(x^*, x^*, x_{n+1}) < \epsilon$ and for the fact that $0 \leq \alpha + 2\beta + \gamma < 1$

Then, $x_{n+1} \to x^*$, $\quad x_n \to x^*$

Hence the sequence converges to x^*.

Next, we prove the fixed point is unique, suppose y^* is another fixed point of T then

$$
\begin{aligned}
S(x^*, x^*, y^*) =& S(Tx^*, Tx^*, Ty^*) \qquad (46) \\
&\leq \alpha \frac{S(x^*, x^*, Tx^*) S(y^*, y^*, Ty^*)}{S(x^*, x^*, y^*)} \\
&+ \beta[S(x^*, x^*, Tx^*) + S(y^*, y^*, Ty^*) \\
&+ \gamma S(x^*, x^*, y^*)] \\
&+ L \min[S(x^*, x^*, Ty^*), S(y^*, y^*, Tx^*)] \\
&\leq \gamma S(x^*, x^*, y^*) + L S(x^*, x^*, y^*) \\
&\leq [\gamma + L] S(x^*, x^*, y^*) < S(x^*, x^*, y^*) \qquad (47)
\end{aligned}
$$

Hence $x^* = y^*$

Therefore, the fixed point is unique

To establish the sequence is data dependence, suppose $U : X \to X$ is an approximate operator of T, then U, will have a fixed point say q^*, if there exists $\eta > 0$ such that

$$S(Tx, Tx, Ux) \leq \eta$$

$$S(x^*, x^*, q^*) \leq t_\eta$$

where, $F_T = \{x^*\}$ $\qquad F_U = q^*$

$$\begin{aligned} S(x^*, x^*, q^*) =& S(Tx^*, Tx^*, Uq^*) && (48)\\ &\leq S(Tx^*, Tx^*, Tq^*) + S(Tq^*, Tq^*, Uq^*) && (49)\\ &\leq \alpha \frac{S(x^*, x^*, Tx^*) S(q^*, q^*, Tq^*)}{S(x^*, x^*, q^*)} \\ &+ \beta[S(x^*, x^*, Tx^*) + S(q^*, q^*, Tq^*) \\ &+ \gamma S(x^*, x^*, q^*)] \\ &+ L \min[S(x^*, x^*, Tq^*), S(q^*, q^*, Tx^*)] + \eta \\ &\leq \beta\gamma S(x^*, x^*, q^*) + LS(x^*, x^*, Tq^*) + \eta && (50)\\ &\leq [\beta\gamma + L] S(x^*, x^*, q^*) + \eta \end{aligned}$$

$$\begin{aligned} S(x^*, x^*, q^*) - [\beta\gamma + L] S(x^*, x^*, q^*) &\leq \eta \\ [1 - \beta\gamma - L] S(x^*, x^*, q^*) &\leq \eta \\ S(x^*, x^*, q^*) &\leq \frac{\eta}{[1 - \beta\gamma - L]} && (51)\\ S(x^*, x^*, q^*) &\leq t_\eta && (52) \end{aligned}$$

The convergence of the fixed points of the two operator depend on $\eta \to 0$. Hence, T and U are data dependence on η. □

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