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# ABOUT THEORY OF PRIMARY DECOMPOSITION OF MONOMIAL IDEAL

## Abstract

In this paper, we have to  $R$  is a commutative Noetherian ring, and we have the  $R$ -module  $I(G)$ , where  $I(G)$  is the edge ideal of a simple and finite graph  $G$ , with no isolated vertices, which is a finitely generated  $R$ -module. We consider also  $\mathfrak{a}$  an ideal of  $R$  and  $N$  a submodule of  $I(G)$  such that  $\mathfrak{a}I(G) \subseteq N$ , an inclusion of modules together with the edge ideal. Here in the article, the edge primary decomposition and irreducible decomposition of  $\mathfrak{a} \times N$  are given.

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## 1. Introduction

Throughout this paper,  $R$  is a commutative ring with non-zero identity.

We consider  $I(G)$ , which is an  $R$ -module, which is the edge ideal of a graph. In the Cartesian product  $R \times I(G)$ , define addition and multiplication as follows:

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1).$$

This is called the *edge idealization* of  $I(G)$ .

In this note, we give an edge primary decomposition of  $\mathfrak{a} \times N$  via edge primary decompositions of  $\mathfrak{a}$  and  $N$ . The edge irreducible decomposition of  $\mathfrak{a} \times N$  is also presented in the end of the paper.

Associated to the graph  $G$  is a monomial ideal

$$I(G) = (v_iv_j \mid v_iv_j \text{ is an edge of } G),$$

with  $v_iv_j = v_jv_i$ ,  $i \neq j$ , in the polynomial ring  $R = K[v_1, v_2, \dots, v_s]$  over a field  $K$ , called the *edge ideal* of  $G$ .

We mean by a graph  $G$ , a finite simple graph with the vertex set  $V(G)$  having no isolated vertex.

In the Section 2, we presented some prerequisites.

In the Section 3, we presented some results about the theory in question. We can consult the references [1] and [5].

We finalize the paper with a conclusion.

We refer to [3, 6] for basics in commutative algebra and homological algebra.

## 2. Prerequisites

This section is in accordance with [2] and [7].

Let  $R = K[v_1, \dots, v_s]$  be a polynomial ring over a field  $K$ , and let  $Z = \{z_1, \dots, z_q\}$  be a finite set of monomials in  $R$ . The *monomial subring* spanned by  $Z$  is the  $K$ -subalgebra,

$$K[Z] = K[z_1, \dots, z_q] \subset R.$$

In general, it is very difficult to certify whether  $K[Z]$  has a given algebraic property - e.g., Cohen-Macaulay, normal - or to obtain a measure of its numerical invariants - e.g., Hilbert function. This arises because the number  $q$  of monomials is usually large.

Thus, consider any graph  $G$ , simple and finite without isolated vertices, with vertex set  $V(G) = \{v_1, \dots, v_s\}$ .

Let  $Z$  be the set of all monomials  $v_i v_j = v_j v_i$ , with  $i \neq j$ , in  $R = K[v_1, \dots, v_s]$ , such that  $\{v_i v_j\}$  is an edge of  $G$ , i.e., the graph finite and simple  $G$ , with no isolated vertices, is such that the squarefree monomials of degree two are defining the edges of the graph  $G$ .

If  $G$  is a graph without isolated vertices, simple and finite, then let  $R$  be denote the polynomial ring on the vertices of  $G$  over some fixed field  $K$ .

We presented now, the definition of the edge ideal of a graph  $G$ , which is finite and simple.

**Definition 2.1** [2]. According to the previous context, the *edge ideal* of a finite simple graph  $G$ , with no isolated vertices, is defined by:

$$I(G) = (v_i v_j \mid v_i v_j \text{ is an edge of } G),$$

with  $v_i v_j = v_j v_i$ , and also with  $i \neq j$ ,

### 3. Main Results about Primary Decomposition

Here, we take  $K$  a fixed field and we consider  $K[v_1, v_2, \dots, v_s]$ , which is the ring polynomial over the field  $K$ .

Since  $K$  is a field, we have that  $K$  is a Noetherian ring and then we have that  $K[v_1, \dots, v_s]$  is also a Noetherian ring (by the Theorem of the Hilbert

Basis).

**Remark 3.1.** By the previous context,  $R = K[v_1, v_2, \dots, v_s]$  is a Noetherian ring. Therefore, the edge ideal  $I(G)$  is a finitely generated ideal. Thus, the edge ideal  $I(G)$  is a finitely generated  $R$ -module, and therefore is a Noetherian  $R$ -module.

The terminology and notations of primary decomposition of ideal or submodule, are found in [4]. We put then, the following definition.

**Definition 3.2.** Let  $R = K[v_1, \dots, v_s]$ . Let  $N$  be an  $R$ -submodule of  $I(G)$ . We define the edge primary decomposition  $N = N_1 \cap \dots \cap N_t$  of  $N$  being *irredundant* or *minimal* if

- (1) the prime ideals  $\sqrt{\text{Ann}_R(I(G)/N_1)}, \dots, \sqrt{\text{Ann}_R(I(G)/N_t)}$  are distinct, and
- (2) for any  $j = 1, \dots, t$ , we have  $N \neq \bigcap_{i \neq j} N_i$ .

Assume, now, that  $\mathfrak{a}$  is an ideal of  $R$ . In the special case for ideals, an edge primary decomposition  $\mathfrak{a} = Q_1 \cap \dots \cap Q_s$  of  $\mathfrak{a}$  is irredundant or minimal if we have  $\sqrt{Q_1}, \dots, \sqrt{Q_s}$  all distinct, and  $\mathfrak{a} \neq \bigcap_{i \neq j} Q_i$ , for any index  $j \in \{1, \dots, s\}$ .

**Remark 3.3.** We have that  $R = K[v_1, \dots, v_s]$  is a Noetherian ring, and that  $I(G)$  is finitely generated. Then there exists a minimal decomposition of  $\mathfrak{a}$  and of  $N$ .

And by [5, page 24], we remark that if  $N$  is an  $\mathfrak{p}$ -primary submodule of  $I(G)$ ,

$$\text{Ann}_R(I(G)/N) \times I(G),$$

is  $\mathfrak{p} \times I(G)$ -primary.

**Definition 3.4.** Let  $R = K[v_1, \dots, v_s]$ . In the context of the Remark 3.3, if  $N$  is an  $\mathfrak{p}$ -primary submodule of  $I(G)$  we said to be  $N$  an edge  $\mathfrak{p}$ -primary submodule of  $I(G)$ , and then we have that

$$\text{Ann}_R(I(G)/N) \times I(G),$$

is an edge  $\mathfrak{p} \times I(G)$ -primary module.

We have now the following result, which we put in the form of a lemma.

**Lemma 3.5.** Let  $R = K[v_1, \dots, v_s]$ . Let  $\mathfrak{a}$  be an ideal of  $R$  and let  $N$  be an  $R$ -submodule of  $I(G)$ . Then,  $\mathfrak{a} \times N$  is edge primary module if and only if either

(1)  $N = I(G)$  and  $\mathfrak{a}$  is an edge primary ideal of  $R$  or

(2)  $N \subset I(G)$  and  $N \neq I(G)$ ,  $\mathfrak{a}I(G) \subseteq N$ , and  $\mathfrak{a}$  and  $N$  are edge  $\mathfrak{p}$ -primary where  $\mathfrak{p} = \sqrt{\mathfrak{a}}$ .

In either cases,  $\mathfrak{a} \times N$  is an edge  $\sqrt{\mathfrak{a}} \times I(G)$ -primary module.

Let  $\mathfrak{a}$  such that  $\mathfrak{a}I(G) \subseteq N$ . We presented now more a result. It is useful to get the edge primary decomposition of the ideal  $\mathfrak{a} \times N$ , which is an  $R$ -module.

**Proposition 3.6.** Let  $R = K[v_1, \dots, v_s]$ . Let  $\mathfrak{a}$  such that  $\mathfrak{a}I(G) \subseteq N$ . Suppose that  $\text{Ass}_R(R/\mathfrak{a}) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$ ,  $\text{Ass}_R(I(G)/N) = \{\mathfrak{q}_1, \dots, \mathfrak{q}_t\}$ , and

$$\text{Ass}_R(R/\mathfrak{a}) \cap \text{Ass}_R(I(G)/N) \neq \emptyset,$$

with  $\mathfrak{p}_i = \mathfrak{q}_i$ , for  $i = 1, \dots, r$ ,  $1 \leq r \leq \min\{s, t\}$ . Then there exists the following minimal edge primary decompositions of  $\mathfrak{a}$  and  $N$ ,

$$\mathfrak{a} = \bigcap_{i=1}^s Q_i; \quad N = \bigcap_{i=1}^t N_i$$

such that  $Q_i I(G) \subseteq N_i$ , for all  $i = 1, \dots, r$ .

**Proof.** Suppose that

$$\mathfrak{a} = \bigcap_{i=1}^s Q'_i; \quad N = \bigcap_{i=1}^t N'_i,$$

are minimal edge primary decompositions of  $\mathfrak{a}$  and  $N$ . Since

$$\mathfrak{p}_1 = \sqrt{\text{Ann}_R(I(G)/N'_1)}, \text{ we have that } N + \mathfrak{p}_1^n I(G) \subseteq N'_1,$$

for  $n$  large enough.

Set  $N_1 = (N + \mathfrak{p}_1^n I(G))_{\mathfrak{p}_1} \cap I(G)$  for  $n$  large enough. We have

$$N \subseteq N_1 \subseteq N'_1,$$

and  $N_1$  is edge  $\mathfrak{p}_1$ -primary submodule of  $I(G)$ . Since

$$N = \bigcap_{i=1}^t N'_i \supseteq N_1 \cap \left( \bigcap_{i=2}^t N'_i \right) \supseteq N,$$

we have that

$$N = N_1 \cap \left( \bigcap_{i=2}^t N'_i \right)$$

is minimal edge primary decomposition of  $N$ . Set  $Q_1 = (\mathfrak{a} + \mathfrak{p}_1^n) R_{\mathfrak{p}_1} \cap R$ .

It is similar as above, we have that

$$\mathfrak{a} = Q_1 \cap \left( \bigcap_{i=1}^t Q'_i \right).$$

Note that  $Q_1 I(G) \subseteq N_1$ .

Moreover, after  $r$  steps, set  $N_i = N'_i$ , for  $i = r + 1, \dots, t$ ,  $Q_i = Q'_i$ , for  $i = r + 1, \dots, s$ . Thus, we have the requirement. This finishes the proof.

**Theorem 3.7.** *Let  $R = K[v_1, \dots, v_s]$ . Let  $\mathfrak{a}$  be such that  $\mathfrak{a}I(G) \subseteq N$ . Set*

$$\Lambda_1 = \{i \mid \mathfrak{p}_i \in \text{Ass}_R(R/\mathfrak{a}) \cap \text{Ass}_R(I(G)/N)\},$$

$$\Lambda_2 = \{i \mid \mathfrak{p}_i \in \text{Ass}_R(R/\mathfrak{a}) \setminus \text{Ass}_R(I(G)/N)\},$$

$$\Lambda_3 = \{i \mid \mathfrak{q}_i \in \text{Ass}_R(I(G)/N) \setminus \text{Ass}_R(R/\mathfrak{a})\}.$$

Assume that

$$\mathfrak{a} = \bigcap_{i=1}^s Q_i, \quad N = \bigcap_{i=1}^t N_i,$$

are minimal edge primary decomposition of  $\mathfrak{a}$  and  $N$  such that  $Q_i I(G) \subseteq N_i$ , for all  $i \in \Lambda_1$ . Then

$$\mathfrak{a} \times N = \bigcap_{i \in \Lambda_1} (Q_i \times N_i) \bigcap_{i \in \Lambda_2} (Q_i \times I(G)) \bigcap_{i \in \Lambda_3} (\text{Ann}_R(I(G)/N_i) \times N_i)$$

is minimal edge primary decomposition of  $\mathfrak{a} \times N$ .

**Proof.** By Lemma 3.5,  $Q_i \times N_i$  is edge  $\mathfrak{p}_i \times I(G)$ -primary module, if  $i \in \Lambda_1$ ,  $Q_j \times I(G)$  is edge  $\mathfrak{p}_j \times I(G)$ -primary module, if  $j \in \Lambda_2$ . Also,  $\text{Ann}_R(I(G)/N_k) \times N_k$  is edge  $\mathfrak{p}_k \times I(G)$ -primary module, if  $k \in \Lambda_3$ . Note that  $\mathfrak{a} \subseteq Q_i$  for all  $i \in \Lambda_1 \cup \Lambda_2$ . By the hypothesis, it follows that  $Q_i I(G) \subseteq N_i$ , for all  $i \in \Lambda_3$ . Then, we have  $\mathfrak{a} \subseteq \text{Ann}_R(I(G)/N_i)$ , for all

$i \in \Lambda_3$ . On the other hand, we have  $N \subseteq N_i$  for all  $i \in \Lambda_1 \cup \Lambda_3$  and  $N \subseteq I(G)$ . Thus

$$\mathfrak{a} \times N \subseteq \bigcap_{i \in \Lambda_1} (Q_i \times N_i) \bigcap_{i \in \Lambda_2} (Q_i \times I(G)) \bigcap_{i \in \Lambda_3} (Ann_R(I(G)/N_i) \times N_i).$$

Conversely, assume that

$$(a, x) \in \bigcap_{i \in \Lambda_1} (Q_i \times N_i) \bigcap_{i \in \Lambda_2} (Q_i \times I(G)) \bigcap_{i \in \Lambda_3} (Ann_R(I(G)/N_i) \times N_i).$$

Then, we have

$$a \in \bigcap_{i \in \Lambda_1 \cup \Lambda_2} Q_i \bigcap_{i \in \Lambda_3} Ann_R(I(G)/N_i) \subseteq \bigcap_{i=1}^s Q_i = \mathfrak{a}.$$

On the other hand,

$$x \in \bigcap_{i \in \Lambda_1 \cup \Lambda_3} N_i \cap I(G) = \bigcap_{i=1}^t N_i = N.$$

This prove that  $(a, x) \in \mathfrak{a} \times N$ . Thus, it follows that the edge primary decompositions of  $\mathfrak{a} \times N$  is minimal. This finishes the proof.

From now on, we assume that  $\dim(I(G)) = t$ .

**Definition 3.8.** Let  $R = K[v_1, \dots, v_s]$ . For an integer  $0 \leq i < t$ , let  $I(G)_i$  denote the largest submodule of  $I(G)$  such that  $\dim_R(I(G)_i) \leq i$ , and  $I(G)_t = I(G)$ . Since  $I(G)$  is a Noetherian  $R$ -module, we have that there exists  $I(G)_i$  and also,  $I(G)_{i-1} \subseteq I(G)_i$ .

The increasing filtration  $\{I(G)_i\}_{0 \leq i \leq t}$  of submodules of  $I(G)$  is called *edge dimension filtration* of  $I(G)$ .

**Lemma 3.9.** Let  $R = K[v_1, \dots, v_s]$ . Assume that

$$0 = \bigcap_{i=1}^t N_i$$

is a minimal edge primary decomposition of 0 in  $I(G)$ . Then

$$I(G)_i = \bigcap_{\dim(R/\mathfrak{p}_j) > i} N_j.$$

The following result give a description of edge dimension filtration of edge idealization of  $I(G)$ .

**Theorem 3.10.** Let  $R = K[v_1, \dots, v_s]$ .

Assume that  $\{R_i\}_{i=0, \dots, d}$  and  $\{I(G)_i\}_{i=0, \dots, t}$  are edge dimension filtration of  $R$  and  $I(G)$ , respectively. Set  $W = R \times I(G)$ ,  $W_i = R_i \times I(G)_i$ , for  $i = 0, \dots, t$ , and  $W_i = R_i \times I(G)$ , for  $i = t + 1, \dots, d$ . Then, we have that  $\{W_i\}_{i=0, \dots, d}$  is the edge dimension filtration of  $W$ .

**Proof.** Set  $\Lambda_i^k = \{i \in \Lambda_i \mid \dim((R \times I(G))/(\mathfrak{p}_i \times I(G))) > k\}$ , for  $i = 1, 2, 3$ . Thus, we have that

$$(R \times I(G))/(\mathfrak{Q}_i \times N_i) \cong R/\mathfrak{Q}_i \times I(G)/N_i$$

and

$$\dim((R \times I(G))/(\mathfrak{Q}_i \times N_i)) = \dim(R/\mathfrak{Q}_i) = \dim(I(G)/N_i)$$

for all  $i \in \Lambda_1$ . Moreover, we have that

$$(R \times I(G))/(\mathfrak{Q}_i \times I(G)) \cong R/\mathfrak{Q}_i \times I(G)$$

and

$$\dim((R \times I(G))/(Q_i \times I(G))) = \dim(R/Q_i)$$

for all  $i \in \Lambda_2$ . Also, we have that

$$(R \times I(G))/(Ann_R(I(G)/N_i) \times N_i) \cong R/Ann_R(I(G)/N_i) \times I(G)/N_i$$

and

$$\begin{aligned} & \dim((R \times I(G))/(Ann_R(I(G)/N_i) \times N_i)) \\ &= \dim(R/Ann_R(I(G)/N_i)) = \dim(I(G)/N_i) \end{aligned}$$

for all  $i \in \Lambda_3$ . So, by Lemma 3.9,

$$W_k = \bigcap_{i \in \Lambda_2^k} (Q_i \times I(G)) = \left( \bigcap_{\dim(R/p_i) > k} Q_i \right) \times I(G) = R_k \times I(G)$$

if  $k = t + 1, \dots, d$  and

$$\begin{aligned} W_k &= \bigcap_{i \in \Lambda_1^k} (Q_i \times N_i) \bigcap_{i \in \Lambda_2^k} (Q_i \times I(G)) \bigcap_{i \in \Lambda_3^k} (Ann_R(I(G)/N_i) \times N_i) \\ &= \left( \bigcap_{\dim(R/p_i) > k} Q_i \right) \times \left( \bigcap_{\dim(R/p_i) > k} N_i \right) = R_i \times I(G)_i \end{aligned}$$

if  $k = 0, \dots, t$ . This completes the proof.

**Lemma 3.11.** *Let  $R = K[v_1, \dots, v_s]$ . Let  $\mathfrak{a}$  be such that  $\mathfrak{a}I(G) \subseteq N$ . Then  $\mathfrak{a} \times N$  is edge irreducible if and only if either*

- (1)  $N = I(G)$  and  $\mathfrak{a}$  is an irreducible ideal of  $R$ , or
- (2)  $N \subseteq I(G)$ , and  $N \neq I(G)$ ,  $\mathfrak{a} = Ann_R(I(G)/N)$ , and  $N$  is irreducible.

**Proof.** (1) It follows by noting that every ideal of  $R \times I(G)$  which

contains  $\mathfrak{a} \times I(G)$  has the form  $J \times I(G)$ , with  $\mathfrak{a} \subseteq J$ .

(2) It follows from [1, Proposition 4.4]. This finishes the proof.

**Theorem 3.12.** *Let  $R = K[v_1, \dots, v_s]$ . Let  $\mathfrak{a}$  be such that  $\mathfrak{a}I(G) \subset N$ .*

*Assume that*

$$\mathfrak{a} = \bigcap_{i=1}^t Q_i; \quad N = \bigcap_{i=1}^r N_i$$

*are edge irreducible decomposition of  $\mathfrak{a}$  and  $N$ . Then*

$$\mathfrak{a} \times N = \bigcap_{i=1}^t (Q_i \times I(G)) \bigcap_{i=1}^r (\text{Ann}_R(I(G)/N_i) \times N_i)$$

*is edge irreducible decomposition of  $\mathfrak{a} \times N$ .*

**Proof.** By Lemma 3.11, we have that  $Q_i \times I(G)$ , is edge irreducible, for  $i = 1, \dots, t$ , and also we have that  $\text{Ann}_R(I(G)/N_i) \times N_i$ , for  $i = 1, \dots, r$ , is edge irreducible. Thus, we have the requirement.

### Conclusion

With the results of this paper, we introduce commutative algebra theory as a useful tool for application in graph theory. And so we provide a relation of the general theory of modules in a particular case, making the necessary considerations, to obtain relevant results within the study area in question.

### Data Availability

No applicable. This manuscript does not report data generation or analysis.

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